BIMETRIC THEORY OF FRACTIONAL QUANTUM HALL STATES

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Chaos, Duality and Topology in CMT, 2017
ACKNOWLEDGMENTS

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Other major references

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PLAN

Introduction to QH effect in curved space

Girvin-MacDonald-Platzman mode

- Lowest Landau Level
- $W_\infty$ algebra
- Single Mode Approximation

Bimetric theory of FQH states

- Bimetric theory
- How does it work?
- Consistency checks

Conclusions and open directions
AT THE QUANTUM HALL PLATEAU

- Gap to all excitations (charged and neutral)
- All dissipative transport coefficients vanish
- Parity and time-reversal broken, but $\mathcal{PT}$-symmetric
- No Lorentz invariance
- Quantized non-dissipative transport coefficients
- Not uniquely characterized by the filling factor

\[ N = \nu N_\phi \quad \quad \sigma_{xy} = \nu \frac{e^2}{h} \]
Does anything universal happen at the scale $E \sim \text{gap}$?

Girvin, MacDonald, Platzman 1986

Balram, Pu 2017

Haldane Rezayi 1985
Geometry is encoded into time-dependent metric

\[ ds^2 = g_{ij}(x, t) dx^i dx^j \]

It is more convenient to use vielbeins

\[ g_{ij} = e_i^A e_j^B \delta_{AB} \]

\[ g = e \cdot e^T \]

There is a \( SO(2) \) redundancy

Corresponding ``gauge field'' is the \textit{spin} connection \( \omega_\mu \)

Spin connection is a ``vector potential'' for curvature

\[ \frac{R}{2} = \partial_1 \omega_2 - \partial_2 \omega_1 \]

\[ \omega_0 \sim \epsilon_A^B e_i^B \partial_0 e_i^A \]
CHERN - SIMONS THEORY OF FQH STATES

\[ S = \frac{k}{4\pi} \int ada - \frac{q}{2\pi} \int adA - \frac{s}{2\pi} \int ad\omega \]

- Determines filling \( \nu = k^{-1} \)
- Electric charge of constituent particles
- "Mean orbital spin"
- Wen-Zee term
- Quantum "emergent" gauge field
- External e/m field
- \( SO(2) \) spin connection

For multi-component states each component has its own \( S_I \)

Wen, Zee 1991
Wen-Zee term couples the electron density to curvature

$$\rho = \frac{\nu}{2\pi} B + \frac{\nu s}{4\pi} R$$

Implies a global relation on a compact Riemann surface

$$N = \nu N_{\phi} + \nu S \frac{\chi}{2}$$

Quantum number $S = 2s$ is called Shift

Also describes the quantum Hall viscosity

$$\langle T_{xx} T_{xy} \rangle = i \omega \eta_H \quad \eta_H = \hbar \frac{S}{4\rho}$$

Beyond TQFT we face a strongly interacting problem

What can we do about it?

• Trial states
• Exact diagonalization
• Hydrodynamics
• Flux attachment (composite bosons and fermions)
• *Bimetric theory*
The GMP mode has been observed in inelastic light scattering experiments.
GENERAL REMARKS ABOUT THE GMP MODE

★ Universally present in **fractional** QH states
★ Absent in **integer** QH states
★ Angular momentum or "spin" 2, regardless of microscopic details
★ Nematic phase transition = condensation of the GMP mode
★ Effective theory of the GMP mode should to be a *theory of massive spin-2 excitation*
The electron density operator

\[
\rho(x) = \sum_{i=1}^{N_{el}} \delta(x - x_i)
\]

Fourier

\[
\rho(k) = \frac{1}{2\pi} \sum_{i=1}^{N_{el}} e^{ik \cdot x_i}
\]

In complex coordinates

\[
k \cdot x_i = \bar{k} z_i + k \bar{z}_i
\]

After the Lowest Landau Level projection

\[
\bar{z} \quad \rightarrow \quad 2\partial_z
\]

Projected density operators

\[
: \bar{\rho}(k) : = \sum_{i=1}^{N_{el}} e^{ik \partial z_i} e^{i\bar{k}z_i}
\]

Satisfy \( W_\infty \) algebra

\[
[\bar{\rho}(k), \bar{\rho}(q)] = 2i \sin \left[ \frac{\ell^2}{2} k \times q \right] \bar{\rho}(k + q)
\]
The LLL generators of $W_\infty$ are $\mathcal{L}_{n,m} = \sum_{i=1}^{N_{el}} z_i^{n+1} \partial z_i^{m+1}$.

Operators $\{\mathcal{L}_{0,0}, \mathcal{L}_{1,-1}, \mathcal{L}_{-1,1}\}$ form $\mathfrak{sl}(2,\mathbb{R})$ algebra.

The projected density operator is expanded in $\mathcal{L}_{n,m}$:

$$\bar{\rho}(k) = e^{-\frac{|k|^2}{2}} \sum_{m,n} c_{nm} \bar{k}^n k^m \mathcal{L}_{n-1,m-1}$$

$\mathcal{L}_{n,m}$ create *intra-LL* state at momentum $k$. 
At long wave-lengths the GMP mode is

\[ \bar{\rho}(k)|0\rangle = \left[ \frac{k^2}{8} L_{-1,1} + \frac{\bar{k}^2}{8} L_{1,-1} + \ldots \right] |0\rangle \]

The GMP state \( \bar{\rho}(k)|0\rangle \) is a shear distortion at small \( k \)

For IQH \( \bar{H} = 0 \) \( \rightarrow \) \( \bar{\rho}(k)|0\rangle \) is a 0 energy state

Consider two-body Hamiltonian \( \bar{H} = \sum_k V(k)\bar{\rho}(-k)\bar{\rho}(k) \)

Since \( [H, \bar{\rho}(k)] \neq 0 \) the shear distortion costs energy

At small \( k \) GMP mode is a gapped, propagating, shear distortion of the FQH fluid
BIMETRIC THEORY
The spin-2 mode is described by a symmetric matrix $\mathbf{h}_{AB}(\mathbf{x}, t)$.

Given $\mathbf{h}_{AB}$ we introduce an ``intrinsic'' metric and vielbein

$$\hat{g}_{ij} = e_i^A e_j^B \mathbf{h}_{AB} = \hat{e}_i^\alpha \hat{e}_j^\beta \delta_{\alpha\beta}$$

FQH constraint: $\sqrt{g} = \sqrt{\hat{g}}$

$SO(2)$ spin connection and curvature follow

$$\frac{\hat{R}}{2} = \partial_1 \hat{\omega}_2 - \partial_2 \hat{\omega}_1 \quad \hat{\omega}_0 = \frac{1}{2} \epsilon_{\alpha\beta} \hat{e}_\alpha^i \partial_0 \hat{e}_\beta^i$$

*Not* the same as two copies of Riemannian geometry

$$\text{Diff} \times \text{Diff} \rightarrow \text{Diff}_{\text{diag}}$$

This geometry involves two metrics $(g_{ij}, \hat{g}_{ij})$, hence *bimetric*

*Appeared recently in theories of massive gravity*
Can visualize $\hat{g}_{i,j}$ as a (fluctuating) pattern on the surface
Chern-Simons theory interacting with fluctuating metric

\[ \mathcal{L} = \frac{k}{4\pi} ada - \frac{1}{2\pi} A da - \frac{s}{2\pi} ad\omega - \frac{s}{2\pi} ad\hat{\omega} + S_{pot}[\hat{g}] \]

We integrate out the gauge field

\[ \mathcal{L} = \mathcal{L}_1[A, g] + \mathcal{L}_{bm}[\hat{g}; A, g] \]

Where \( \mathcal{L}_1[A, g] \) contains no dynamics and

\[ \mathcal{L}_{bm} = \frac{\nu s}{2\pi} A d\hat{\omega} - \frac{M}{2} \left( \frac{1}{2} \hat{g}_{ij} g^{ij} - \gamma \right)^2 \]

For IQH \( k = 1 \) there is no \textit{intra-LL dynamics} and

\[ \mathcal{L} = \mathcal{L}_1[A; g] \]
Density and current operators acquire geometric meaning

Fluctuations of electron density = fluctuations of local Ricci curvature

\[ \rho = \frac{\nu \varsigma}{4\pi} \hat{R} \]

Fluctuations of electron current = fluctuations of \``gravi-electric`` field

\[ j^i = \frac{\nu \varsigma}{2\pi} \epsilon^{ik} \hat{E}_k \]

To the leading order in \( k \), \textit{everything} is determined by \( \varsigma \)

Continuity equation holds identically

\[ \partial_0 \hat{R} + \epsilon^{ik} \partial_i \hat{E}_k \equiv 0 \]
This potential has two phases

If $\gamma < 1$ the theory is in the gapped "symmetric" phase

$$h_{AB} = \delta_{AB}, \quad \hat{g}_{ij} = g_{ij}$$

If $\gamma > 1$ the theory is in the gapless nematic phase

$$h_{AB} = h_{AB}^{(0)}\quad \hat{g}_{ij} \neq g_{ij}$$

We will be interested in the "symmetric" phase
In flat space we chose the parametrization

\[ h_{AB} = \exp \left( \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & -Q_2 \end{pmatrix} \right), \quad Q = Q_1 + iQ_2 \]

\[ \bar{Q} = Q^* \]

and linearize in flat space around

\[ Q = 0, \quad h_{AB} = \delta_{AB} \]

Gap of the GMP mode

\[ \mathcal{L}_{bm} \approx i \frac{\mathcal{S} \rho_0}{4} \bar{Q} \dot{Q} - \frac{m}{2} |Q|^2 \]
To determine the coefficient $\varsigma$ we calculate the SSF

$$\bar{s}(k) = 2\pi \ell^2 \nu^{-1} \langle \rho_{-k} \rho_k \rangle$$

Calculation in the linearized theory reveals

$$\bar{s}(k) = \frac{2|\varsigma|}{8} |k|^4 + \ldots$$

Match this to a general LLL result for chiral states

$$\bar{s}(k) = \frac{|S - 1|}{8} |k|^4 + \ldots$$

This uniquely determines

$$2|\varsigma| = |S - 1|$$

Vanishes for IQH
From the action we read the CCR

\[ \frac{\nu \varsigma}{2\pi} A d\hat{\omega} = \frac{\nu \varsigma}{2\pi} B \hat{\omega}_0 = \frac{\varsigma \rho_0}{2} \epsilon_{\alpha \beta} \hat{e}^i_{\alpha} \frac{\partial}{\partial t} \hat{e}^i_{\beta} \]

Which leads to the \( \mathfrak{sl}(2, \mathbb{R}) \) algebra for the metric

\[ [\hat{g}_{zz}(x), \hat{g}_{\bar{z}\bar{z}}(x')] = \frac{16}{\rho_0 \varsigma} \hat{g}_{zz}(x) \delta(x - x') \]

\[ [\hat{g}_{\bar{z}\bar{z}}(x), \hat{g}_{zz}(x')] = \frac{8}{\rho_0 \varsigma} \hat{g}_{\bar{z}\bar{z}}(x) \delta(x - x') \]

Turn off external fields

\[ \frac{\nu \varsigma}{2\pi} A d\hat{\omega} = \frac{\nu \varsigma}{2\pi} B \hat{\omega}_0 = \frac{\varsigma \rho_0}{2} \epsilon_{\alpha \beta} \hat{e}^i_{\alpha} \frac{\partial}{\partial t} \hat{e}^i_{\beta} \]

Invariant under \( SL(2, \mathbb{R}) \)

Potential breaks \( SL(2, \mathbb{R}) \)

Appeared in Verlinde 1989
Algebra of the spin connections closes

$$[\hat{\omega}_i(k), \hat{\omega}_j(q)] = \frac{1}{\rho_0\varsigma} \left[ k_j \hat{\omega}_i(k + q) - q_i \hat{\omega}_j(k + q) \right] - \frac{i\epsilon_{ij}}{2\rho_0\varsigma} \hat{R}(k + q)$$

Spin connection couples like the dipole moment

$$\hat{\omega}dA \approx E_i\epsilon_{ij}\hat{\omega}_j = \mathbf{E} \cdot (\epsilon\hat{\omega})$$

(compare to \( U = -\mathbf{E} \cdot \mathbf{d} \))

``Dipole'' algebra implies

$$[\hat{R}(k), \hat{R}(q)] = \frac{4\pi}{\nu\varsigma} i(k \times q)\ell^2 \hat{R}(k + q)$$

Small \( k \) GMP algebra follows

$$[\hat{\rho}(k), \hat{\rho}(q)] \approx il^2(k \times q) \hat{\rho}(k + q)$$
Complete Lagrangian up to three derivatives

\[ \mathcal{L}_{\text{bm}} = \frac{\nu_S}{2\pi} A d\omega - \frac{c}{4\pi} \omega d\omega - \frac{\nu_S}{4\pi} \nabla_i E_i B - \frac{\ell^2}{8\pi} \nabla_i E_i R - \frac{m}{2} \left( \frac{1}{2} g_{ij} g^{ij} - \gamma \right)^2 - \frac{\alpha}{4} \left| \Gamma - \hat{\Gamma} \right|^2 \]

★ \textit{Projected} static structure factor up to \( |k|^6 \)

★ Dispersion relation of the GMP mode up to \( |k|^2 \)

★ \textit{Absence} of the GMP mode and nematic transition in IQH

★ Hidden LLL projection and manifest Particle-Hole duality

★ Girvin-MacDonald-Platzman algebra holds up to \( |k|^4 \)

★ \textsl{``Guiding center''} DC Hall conductivity to \( |k|^2 \)

★ \textsl{``Guiding center''} Hall viscosity to \( |k|^2 \)

★ Shear modulus of the FQH fluid

★ Hints at rich structure of the full \( W_\infty \) theory and more…

\textsuperscript{AG, Son 2017}
OPEN PROBLEMS

★ Understand the non-linearity
★ Fully covariant formulation
★ Implications for the boundary theory
★ CFT construction of the GMP state?
★ Competing orders in multi-layer states
★ Non-linear higher spin theory
★ Fractional Chern insulators
★ Collective neutral fermion mode in 5/2 state
★ Bimetric theory for PH-Pfaffian
★ Covariant, nonlinear formulation of CFL
★ Quantum Hall liquid crystal phases
★ Detailed study of anisotropic FQH states
★ Relation to ``fracton'' theories?
★ 3D
★ .......

Bi-layer FQH

Neutral Fermion in 5/2 state

FCI
``quantum metric''

FQH liquid crystal
SINGLE MODE APPROXIMATION (SMA)

SMA states that observables are saturated by $\bar{\rho}(k)|0\rangle$

For example, optical absorption spectrum

$$\Delta(k) = \frac{\langle 0|\bar{\rho}(-k)H\bar{\rho}(k)|0\rangle}{\langle 0|\bar{\rho}(-k)\bar{\rho}(k)|0\rangle}$$

SMA is accurate at small $k$

SMA is exact near the nematic phase transition
Aharonov-Bohm phases
quantum numbers of quasiholes
Mutual statistics*
Chiral edge modes:
tunneling exponents, thermal Hall conductance

Linear response:
Hall conductance, Hall viscosity, …

Ground state degeneracy

\[ W[A, \omega] = \int Dae^{iS[a;A,\omega]} \]
Electrically charged particles in magnetic field have AB effect

\[ \Psi \rightarrow \exp \left( 2\pi ie \oint_C A_i dx^i \right) \Psi \]
\[ \partial_1 A_2 - \partial_2 A_1 = B \]

Particles with orbital spin in curved space have AB effect

\[ \Psi \rightarrow \exp \left( 2\pi i \bar{s} \oint_C \omega_i dx^i \right) \Psi \]
\[ \partial_1 \omega_2 - \partial_2 \omega_1 = R/2 \]

AHARONOV - BOHM EFFECT

Wen Zee 1992
In the remainder of the talk I will use the term “orbital spin”.

In magnetic field electrons quickly move in cyclotron orbits

\[ \omega_c = \frac{B}{m_{el}} \]

We consider the limit \( m_{el} \rightarrow 0 \)

“Orbital spin” describe the coupling of the low energy physics to spatial geometry.
COMPOSITE FERMI LIQUID

States at filling \( \nu = \frac{N}{2N+1} \approx \) IQH of composite fermions at \( \nu_{\text{eff}} = N \)

Can be treated via Fermi liquid theory when \( N \) is large

Semiclassically the d.o.f. are multipolar distortions of the Fermi surface

- \( \mathbf{u}_0 \) (Dilation)
- \( \mathbf{u}_\pm 1 \) (Translation)
- \( \mathbf{u}_\pm 2 \) (Shear)
- \( \mathbf{u}_\pm 3 \)
- \( \mathbf{u}_\pm 4 \)
- \( \cdots \) (``Higher spin'' area preserving deformations)
- \( \mathbf{u}_\pm n \) (dynamical)
**COMPOSITE FERMI LIQUID IN SMA**

**Hamiltonian**

\[
H = \frac{v_F k_F}{4\pi} \sum_n \int d^2x (1 + F_n) u_n(x) u_{-n}(x),
\]

**CCR**

\[
[u_n(x), u_m(x')] = \frac{2\pi}{k_F^2} \left(n\bar{b}\delta_{n+m,0} - ik_F\delta_{n+m,1}\partial_z - ik_F\delta_{n+m,-1}\partial_{\bar{z}}\right)\delta(x - x')
\]

All modes are gapped at \( \Delta_n = n(1 + F_n)\omega_c \)

The limit \( \Delta_2 \ll \Delta_n \) for all \( n \geq 3 \) is the SMA

Only dynamics of shear distortions \( u_{\pm 2} \) remains

\[
[u_2(x), u_{-2}(x')] = \frac{4\pi}{2N + 1} \delta(x - x')
\]

**Phenomenological \"Landau parameters\"**

Nguyen, AG, Son In Progress
Effective Lagrangian in SMA

$$\mathcal{L}_{\text{SMA}} = -\frac{i}{2} \frac{2N+1}{2\pi} u_2 \dot{u}_{-2} + \frac{i}{2} \frac{N^2(2N+3)\ell^2}{12\pi} u_2 \Delta \dot{u}_{-2} - \frac{c_0(2N+1)\omega_c}{2\pi} u_2 u_{-2} + \frac{c_2(2N+1)\omega_c\ell^2}{2\pi} u_2 \Delta u_{-2}$$

coincides with the linearized bimetric theory

$$\mathcal{L}_{\text{bm}} = \frac{2N+1}{16\pi} \hat{A} d\hat{\omega} - \frac{N^2(2N+3)}{96\pi} \hat{\omega} d\hat{\omega} - \frac{\tilde{m}}{2} \left[ \hat{g}_{ij} \hat{g}^{ij} - \gamma \right]^2 - \frac{\alpha}{4} \left[ \Gamma - \hat{\Gamma} \right]^2$$

Bimetric theory prescribes coupling of the CFL to curved space

Conjecture:

Bimetric theory is the geometric non-linear completion of the CFL in the SMA.

What about beyond SMA?