Classify FQH states through pattern of zeros

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- Here we try to classify the chiral topological orders in FQH states by classifying symmetric polynomials of infinite variable.

Three FQH states

FQH state in the first Landau level (bosonic electrons)

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- $\nu = 1$ Pfaffian state Moore & Read, 1991

$$\Phi_{1/2} = \mathcal{A}\Big(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N}\Big) \prod_{i<j} (z_i - z_j)$$

$$V_{Pf}(z_1, z_2, z_3) = \mathcal{S}[v_0 \delta(z_1 - z_2)\delta(z_2 - z_3) - v_1 \delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial_{z_3}$$

Pattern of zeros

Let $z_i = \lambda\xi_i + z^{(a)}$, $i = 1, 2, \cdots, a$

$$\Phi(\{z_i\}) = \lambda^{S_a} P(\xi^1, ..., \xi^a; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$

- The sequence of integers $\{S_a\}$ characterizes the polynomial $\Phi(\{z_i\})$ and is called the pattern of zeros.
- $\nu = 1/2$ Laughlin state $S_1, S_2, \cdots :\ 0, 2, 6, 12, 20, 30, 42, 56, \cdots$.

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- **Unique fusion cond.**: P does not depend on the "shape" $\{\xi^i\}$

$$P(\{\xi^i\}; z^{(a)}, z_{a+1}, z_{a+2}, \cdots) \propto P(z^{(a)}, z_{a+1}, z_{a+2}, \cdots)$$

- **Pattern of zeros and orbital/occupation distribution**

  Let $l_a = S_a - S_{a-1}$ or $S_a = \sum_{i=1}^{a} l_i$, then

$$\Phi(\{z_i\}) \sim \mathcal{S}[z_1^{l_1} z_2^{l_2} \cdots] + \cdots, \qquad l^{\text{th}} \text{ orbital} = z^l$$

  The pattern of zero of $\nu = 1/2$ Laughlin state is also described by

$$l_1, l_2, \cdots : 0, 2, 4, 6, 8, 10, \cdots$$

$$n_0 n_1 n_2 \cdots : 1010101010101010 \cdots$$

- $\nu = 1/4$ Laughlin state

$$S_1, S_2, \cdots : 0, 4, 12, 24, 40, 60, 84, \cdots$$
$$l_1, l_2, \cdots : 0, 4, 8, 12, 16, 20, \cdots$$
$$n_0 n_1 n_2 \cdots : 10001000100010001 \cdots$$

A cluster (unit cell): 1 particles 4 orbitals

- $\nu = 1$ Pfaffian state

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$
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- FQH <=> 1D "CDW" (on thin cylinder)

Haldane & Rezayi, 94; Seidel & Lee, 06; Bergholtz, Kailasvuori, Wikberg, Hansson, Karlhede, 06; Bernevig & Haldane, 07

A classification problem

- We have seen that
  each symmetric polynomial $\Phi(\{z_i\}) \to \{S_a\}$ a pattern of zeros.
  But each sequence of integers $\{S_a\} \not\to \Phi(\{z_i\})$
- Find all the conditions a sequence $\{S_a\}$ must satisfy, such that $\{S_a\}$ describe a symmetric polynomial that satisfies the unique fusion condition. $\to$
  A classification of symmetric polynomials (FQH states) through pattern of zeros.

Derived polynomials

- Let $z_1, ..., z_a \to z^{(a)}$

$$\Phi(\{z_i\}) = \lambda^{S_a} P(z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$

we get a derived polynomial $P(z^{(a)}, z^{(b)}, z^{(c)}, \cdots)$.

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- Zeros in derived polynomials $D_{a,b}$

$$P(z^{(a)}, z^{(b)}, z^{(c)}, \cdots) \sim (z^{(a)} - z^{(b)})^{D_{a,b}} P'(z^{(a+b)} ...) + \cdots$$

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  also characterize the pattern of zeros.
- The data $D_{a,b}$ and S_a are related:

$$D_{a,b} = S_{a+b} - S_a - S_b.$$

Conditions on pattern of zeros – ground state

- Concave conditions

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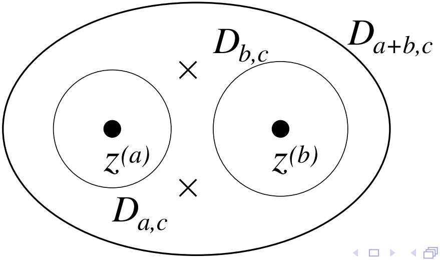
The second one comes from

$$D_{a+b,c} \geq D_{a,c} + D_{b,c}$$

which can be shown by considering $P(z^{(a)}, z^{(b)}, z^{(c)}, \cdots)$ as a function of $z^{(c)}$



- n-cluster condition: No off-particle zeros when $c = n$ (or the wave function for the n-clusters is the Laughlin wave function) $\rightarrow D_{a+b,n} = D_{a,n} + D_{b,n} \rightarrow$

$$S_{a+kn} = S_a + kS_n + \frac{k(k-1)nm}{2} + kma$$

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- Additional conditions

$$\Delta_2(a,a) = \text{even}, \qquad m > 0, \qquad mn = \text{even}, \qquad 2S_n = 0 \text{ mod } n.$$

- A mysterious condition (the one we want but cannot prove):

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- $(m; S_2, \cdots, S_n)$ that satisfy the above conditions correspond to symmetric polynomials. => Those $(m; S_2, \cdots, S_n)$ "classify" symmetric polynomials and FQH states (with $\nu = n/m$).

Primitive solutions for pattern of zeros

The conditions are semi-linear $\to$
if $(m; S_2, \cdots, S_n)$ and $(m'; S'_2, \cdots, S'_n)$ are solutions, then $(m''; S''_2, \cdots, S''_n) = (m; S_2, \cdots, S_n) + (m'; S'_2, \cdots, S'_n)$ is also a solution $\sim$ $\Phi'' = \Phi\Phi'$

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1-cluster state: $\nu = 1/m$ Laughlin state

$$\Phi_{1/m}: \quad \mathbf{S} = (m;),$$
$$(n_0, \cdots, n_{m-1}) = (1, 0, \cdots, 0).$$

2-cluster state: Pfaffian state (Z_2 parafermion state)

$$\Phi_{\frac{2}{2};Z_2}: \quad (m; S_2) = (2; 0),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0)$$

3-cluster state: Z_3 parafermion state

$$\Phi_{\frac{3}{2};Z_3}: \quad (m; S_2, S_3) = (2; 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (3, 0)$$

4-cluster state: Z_4 parafermion state

$$\Phi_{\frac{4}{2};Z_4} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0),$$

$$(n_0, \cdots, n_{m-1}) = (4, 0),$$

5-cluster states: Z_5 (generalized) parafermion state

$$\Phi_{\frac{5}{2};Z_5} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0),$$

$$(n_0, \cdots, n_{m-1}) = (5, 0)$$

$$\Phi_{\frac{5}{8};Z_5^{(2)}} : (m; S_2, \cdots, S_n) = (8; 0, 2, 6, 10),$$

$$(n_0, \cdots, n_{m-1}) = (2, 0, 1, 0, 2, 0, 0, 0)$$

6-cluster state:

$$\Phi_{\frac{6}{2};Z_6} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0, 0),$$

$$(n_0, \cdots, n_{m-1}) = (6, 0)$$

7-cluster states:

$$\Phi_{\frac{7}{2};Z_7} : (m; S_2, \cdots, S_n) = (2; 0, 0, 0, 0, 0, 0),$$
$$(n_0, \cdots, n_{m-1}) = (7, 0)$$

$$\Phi_{\frac{7}{8};Z_7^{(2)}} : (m; S_2, \cdots, S_n) = (8; 0, 0, 2, 6, 10, 14),$$
$$(n_0, \cdots, n_{m-1}) = (3, 0, 1, 0, 3, 0, 0, 0)$$

$$\Phi_{\frac{7}{18};Z_7^{(3)}} : (m; S_2, \cdots, S_n) = (18; 0, 4, 10, 18, 30, 42),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0, 0)$$

$$\Phi_{\frac{7}{14};C_7} : (m; S_2, \cdots, S_n) = (14; 0, 2, 6, 12, 20, 28),$$
$$(n_0, \cdots, n_{m-1}) = (2, 0, 1, 0, 1, 0, 1, 0, 2, 0, 0, 0, 0, 0)$$

- Also get composite parafermion state $\Phi = \Phi_{Z_{n_1}} \Phi_{Z_{n_2}}$

Topological properties from pattern of zeros

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- Number of quasiparticle types (topological degeneracy on torus)
- Quasiparticle charges
- Quasiparticle fusion algebra
- The corresponding CFT (chiral vertex algebra)

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$$\Phi_\gamma(\{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(z^{(a)}, z_{a+1}, z_{a+2}, \cdots) + O(\lambda^{S_a+1})$$

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- $\{S_a\}$ correspond to the trivial quasiparticle $\gamma = 0$: $\{S_{0;a}\} = \{S_a\}$
- To find the allowed quasiparticles, we simply need to find
  (i) the conditions that $S_{\gamma;a}$ must satisfy and
  (ii) all the $S_{\gamma;a}$ that satisfy those conditions.

Conditions on $S_{\gamma;a}$

- Concave condition

$$S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0,$$
$$S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0$$

- n-cluster condition

$$S_{\gamma;a+kn} = S_{\gamma;a} + k(S_{\gamma;n} + ma) + mn\frac{k(k-1)}{2}$$

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- Find all $(S_{\gamma;1}, \cdots, S_{\gamma;n})$ that satisfy that above conditions $\rightarrow$ obtain all the quasiparticles.

For the $\nu = 1$ Pfaffian state ($n = 2$ and $m = 2$)

$$S_1, S_2, \cdots : 0, 0, 2, 4, 8, 12, 18, 24, \cdots$$

$$n_0 n_1 n_2 \cdots : 20202020202020202 \cdots$$

- Quasiparticle solutions:

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 20202020202020202 \cdots \qquad Q_\gamma = 0$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 02020202020202020 \cdots \qquad Q_\gamma = 1$$

$$n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \cdots : 11111111111111111 \cdots \qquad Q_\gamma = 1/2$$

Unit cell: m orbitals + n electrons

- All other quasiparticle solutions can obtained from the above three by removing extra electrons → only 3 quasiparticle types.

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- Ground state degeneracy on torus = number of quasiparticle types
- Charge of quasiparticles

$$Q_\gamma = \frac{1}{m} \sum_{a=1}^{n} (l_{\gamma;a} - l_a)$$

Pattern of zeros and generalized Pauli exclusion rule

In terms of $l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}$, the concave condition for quasiparticles becomes

$$\sum_{k=1}^{b} l_{\gamma;a+k} \geq S_b,$$

$$\sum_{k=1}^{c} (l_{\gamma;a+b+k} - l_{\gamma;a+k}) \geq S_{b+c} - S_b - S_c = D_{b,c}$$

for any $a, b, c \in Z_+$.
Setting $c = 1$: b electrons must spread over $D_{b,1} + 1$ orbitals or more.

Quasiparticle solutions (for states related to known CFT)

For the parafermion states $\Phi_{\nu=\frac{n}{2};Z_n}$ $(m=2)$,

$\Phi_{\frac{2}{2};Z_2}$	$\Phi_{\frac{3}{2};Z_3}$	$\Phi_{\frac{4}{2};Z_4}$	$\Phi_{\frac{5}{2};Z_5}$	$\Phi_{\frac{6}{2};Z_6}$	$\Phi_{\frac{7}{2};Z_7}$	$\Phi_{\frac{8}{2};Z_8}$	$\Phi_{\frac{9}{2};Z_9}$	$\Phi_{\frac{10}{2};Z_{10}}$
3	4	5	6	7	8	9	10	11

For the parafermion states $\Phi_{\nu=\frac{n}{2+2n};Z_n}$ $(m=2+2n)$

$\Phi_{\frac{2}{6};Z_2}$	$\Phi_{\frac{3}{8};Z_3}$	$\Phi_{\frac{4}{10};Z_4}$	$\Phi_{\frac{5}{12};Z_5}$	$\Phi_{\frac{6}{14};Z_6}$	$\Phi_{\frac{7}{16};Z_7}$	$\Phi_{\frac{8}{18};Z_8}$	$\Phi_{\frac{9}{20};Z_9}$	$\Phi_{\frac{10}{22};Z_{10}}$
9	16	25	36	49	64	81	100	121

For the generalized parafermion states $\Phi_{\nu=\frac{n}{m};Z_n^{(k)}}$

$\Phi_{\frac{5}{8};Z_5^{(2)}}$	$\Phi_{\frac{5}{18};Z_5^{(2)}}$	$\Phi_{\frac{7}{8};Z_7^{(2)}}$	$\Phi_{\frac{7}{22};Z_7^{(2)}}$	$\Phi_{\frac{7}{18};Z_7^{(3)}}$	$\Phi_{\frac{7}{32};Z_7^{(3)}}$	$\Phi_{\frac{8}{18};Z_8^{(3)}}$	$\Phi_{\frac{9}{8};Z_9^{(2)}}$
24	54	32	88	72	128	81	40

where k and n are coprime.

For the composite parafermion states $\Phi_{\frac{n_1}{m_1};Z_{n_1}^{(k_2)}}\Phi_{\frac{n_2}{m_2};Z_{n_2}^{(k_2)}}$ obtained as products of two parafermion wave functions

$\Phi_{\frac{2}{2};Z_2}\Phi_{\frac{3}{2};Z_3}$	$\Phi_{\frac{3}{2};Z_3}\Phi_{\frac{4}{2};Z_4}$	$\Phi_{\frac{2}{2};Z_2}\Phi_{\frac{5}{2};Z_5}$	$\Phi_{\frac{2}{2};Z_2}\Phi_{\frac{5}{8};Z_5^{(2)}}$
30	70	63	117

where n_1 and n_2 are coprime. The inverse filling fractions of the above composite states are $\frac{1}{\nu} = \frac{1}{\nu_1} + \frac{1}{\nu_2} = \frac{m_1}{n_1} + \frac{m_2}{n_2}$.

- Those results from the pattern of zeros all agree with the results from parafermion CFT: Barkeshli & Wen, 2008

$$\# \text{ of quasiparticles} = \frac{1}{\nu}\prod_i \frac{n_i(n_i+1)}{2}$$

for the generalized composite parafermions state

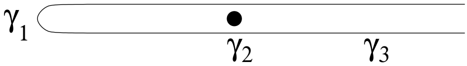
$$\Phi = \prod_i \Phi_{\frac{n_i}{m_i};Z_{n_i}^{(k_i)}}, \quad \{n_i\} \text{ coprime}, \quad (k_i, n_i) \text{ coprime}.$$

$$1/\nu = \sum m_i/n_i$$

Quasiparticle fusion algebra: $\gamma_1\gamma_2 = \sum_{\gamma_3} N^{\gamma_3}_{\gamma_1\gamma_2}\gamma_3$

Consider a particular fusion channel $\gamma_1\gamma_2 \to \gamma_3$. Its occupation representation is a "domain wall" Ardonne etc, 2008

$$n_{\gamma_1;0}n_{\gamma_1;1}\cdots n_{\gamma_1;a}[\gamma_2]n_{\gamma_3;a+1}n_{\gamma_3;a+2}\cdots$$



From the domain wall, we can see $n_{\gamma_1;l}$ and $n_{\gamma_3;l}$, but we do not know $n_{\gamma_2;l}$.

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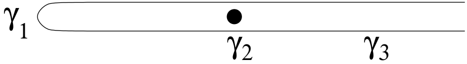

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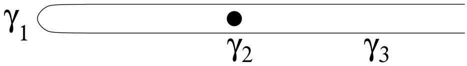

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$$\sum_{j=1}^{b} l^{\text{sc}}_{\gamma_2+c;j} \le \sum_{j=1}^{b} \left(l^{\text{sc}}_{\gamma_3+a+c;j} - l^{\text{sc}}_{\gamma_1+a;j} + l^{\text{sc}}_{j} \right)$$

for any $a, b, c \in Z_+$, where $l^{\text{sc}}_{\gamma;a} = l_{\gamma;a} - \frac{m(Q_\gamma + a - 1)}{n}$

- The condition only determine when $N_{\gamma_1\gamma_2}^{\gamma_3} \neq 0$. If we assume $N_{\gamma_1\gamma_2}^{\gamma_3} = 0, 1$, then the fusion algebra is fixed.
- For generalized composite parafermion states, the pattern-of-zeros approach and the CFT approach give rise to the same fusion algebra.
- The pattern-of-zeros approach applies to other FQH states whose CFT may not be known.

Summary

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- Pattern of zeros
- Tensor category theory
- Conformal field theory (chiral algebra)

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Pattern of long range entanglement

Mathematical foundation of topological/quantum orders