In search of topological states with half quantum vortices.

Eun-Ah Kim
Cornell University

Suk-Bum Chung (Stanford) Hendrik Bluhm (Harvard)
Subroto Mukerjee (Berkeley) Daniel Agterberg (UWM)
In search of topological states with half quantum vortices

- Topological order and fractionalization
- 1/2 QV’s
- Stability of 1/2-QV’s in SrRuO
- 1/2 QV lattices
Ground state degeneracy
Ground state degeneracy

Conventional order
Ground state degeneracy

Conventional order
Ground state degeneracy

Conventional order
Ground state degeneracy

Conventional order

Symmetry of the underlying Hamiltonian.
Ground state degeneracy

Conventional order

Symmetry of the underlying Hamiltonian.

\( \Leftrightarrow \text{reduced symmetry} \)
Ground state degeneracy

Conventional order

Symmetry of the underlying Hamiltonian.

\[ \text{reduced symmetry} \]

Local measurements

\[ \text{order parameter} \]
Ground state degeneracy

Conventional order

Symmetry of the underlying Hamiltonian.

减少了的对称性

Topological order

Local measurements

有序参数
Ground state degeneracy

**Conventional order**
- Symmetry of the underlying Hamiltonian.
- Reduced symmetry
- Local measurements
- Order parameter

**Topological order**
- Gapped spectrum
Ground state degeneracy

Conventional order vs Topological order

Symmetry of the underlying Hamiltonian.

- Reduced symmetry

Local measurements

- Order parameter

Gapped spectrum

No local order parameter.
Ground state degeneracy

Conventional order

- Symmetry of the underlying Hamiltonian.
- \textbf{reduced symmetry}
- Local measurements
- \textbf{order parameter}

Topological order

- Gapped spectrum
- No local order parameter.
- Topological degeneracy $N_g$. 
Ground state degeneracy

Conventional order

- Symmetry of the underlying Hamiltonian.
- Local measurements

Topological order

- Gapped spectrum
- No local order parameter.
- Topological degeneracy $N_g$. 

*reduced symmetry*
Ground state degeneracy

Conventional order
- Symmetry of the underlying Hamiltonian.
- Reduced symmetry
- Local measurements
  ↔ order parameter

Topological order
- Gapped spectrum
- No local order parameter.
- Topological degeneracy $N_g$. 
Ground state degeneracy

Conventional order

- Symmetry of the underlying Hamiltonian.
- Reduced symmetry
- Local measurements
- Order parameter

Topological order

- Gapped spectrum
- No local order parameter.
- Topological degeneracy $N_g$. 
Ground state degeneracy

Conventional order

Symmetry of the underlying Hamiltonian.
↔ reduced symmetry

Local measurements
↔ order parameter

Topological order

Gapped spectrum

Topological invariance
↔ emergent symmetry

No local order parameter.

Topological degeneracy $N_g$. 
Sweet Topology
Sweet Topology
Sweet Topology
Fractional charge $e^* = e/q$

$N_g = q^g \text{ e.g., } N_1 = 3$

Wen and Niu, PRB, 1990
Stone and Chung, PRB, 2006
Hansson, Oganesyan, Sondhi, Ann.Phys, 2004
\( N_g \) & fractionalization

- Fractional charge \( e^* = e/q \)
  - \( N_g = q^g \) e.g., \( N_1 = 3 \)

- 2n Non-abelian vortices

Wen and Niu, PRB, 1990
Stone and Chung, PRB, 2006
Hansson, Oganesyan, Sondhi
Ann.Phys, 2004
$N_g$ & fractionalization

Fractional charge $e^* = e/q$

$N_g = q^g$, e.g., $N_1 = 3$

2n Non-abelian vortices

Wen and Niu, PRB, 1990
Stone and Chung, PRB, 2006
Hansson, Oganesyan, Sondhi, Ann.Phys, 2004
\( N_g \) & fractionalization

\( \text{\ding{52} Fractional charge } e^* = \frac{e}{q} \)

\( N_g = q^g \text{ e.g., } N_1 = 3 \)

\( \text{\ding{53} 2n Non-abelian vortices} \)

\( N_{2n} = 2^{n-1} \text{ for MR state or } p+ip \text{ SF} \)

Wen and Niu, PRB, 1990
Stone and Chung, PRB, 2006
Hansson, Oganesyan, Sondhi Ann.Phys, 2004
N_g & fractionalization

- Fractional charge $e^* = e/q$
- $N_g = q^g$ e.g., $N_1 = 3$

- 2n Non-abelian vortices
- $N_{2n} = 2^{n-1}$ for MR state or p+ip SF

References:
- Wen and Niu, PRB, 1990
- Stone and Chung, PRB, 2006
- Hansson, Oganesyan, Sondhi, Ann.Phys, 2004
Abelian statistics
Abelian statistics

n- *abelian* vortex states
Abelian statistics

$n$-abelian vortex states

$$\Psi(x_1, \cdots, x_n) = c\text{-number}$$
Abelian statistics

\( n \) - abelian vortex states

\[ \Psi(x_1, \ldots, x_n) = c\text{-number} \]

exchange of qp’s:

phase multiplication to a complex number

\[ \Psi(x_1 \leftrightarrow x_3) = e^{i\theta} \Psi \]
\[ \Psi(x_1 \leftrightarrow x_2) = e^{i\theta} \Psi \]
Nonabelian statistics
Nonabelian statistics

$n$-nonabelian vortex states $\Rightarrow$ set of Qubits
Nonabelian statistics

$n$-nonabelian vortex states $\Rightarrow$ set of Qubits

$$\Psi(x_1, \cdots, x_n) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d(n)} \end{pmatrix}$$
Nonabelian statistics

Nonabelian vortex states \( \Rightarrow \) set of Qubits

\[
\Psi(x_1, \ldots, x_n) = \begin{pmatrix}
\psi_1 \\
\vdots \\
\psi_d(n) \\
\end{pmatrix}
\]
Nonabelian statistics

\( n - \text{nonabelian} \) vortex states \( \Rightarrow \) set of Qubits

\[ \Psi(x_1, \ldots, x_n) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d(n)} \end{pmatrix} \]

exchange of qp’s: rotation in \( d(n) \) dim Hilbert space
Nonabelian statistics

\( \text{nonabelian vortex states} \rightarrow \text{set of Qubits} \)

\[ \Psi(x_1, \cdots, x_n) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d(n)} \end{pmatrix} \]

exchange of qp's:

rotation in \( d(n) \) dim Hilbert space

\[ \Psi(x_1 \leftrightarrow x_3) = M \Psi(x_1, \cdots, x_n) \]

\[ \Psi(x_1 \leftrightarrow x_2) = N \Psi(x_1, \cdots, x_n) \]
Nonabelian statistics

\[ \Psi(x_1, \ldots, x_n) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d(n)} \end{pmatrix} \]

exchange of qp's:

rotation in \(d(n)\) dim Hilbert space

\[ \Psi(x_1 \leftrightarrow x_3) = M \Psi(x_1, \ldots, x_n) \]

\[ \Psi(x_1 \leftrightarrow x_2) = N \Psi(x_1, \ldots, x_n) \]

\(d(2n) = 2^{n-1}\) for MR state or p+ip SC
In search of topological states with fractionalized excitations.

- Topological order and fractionalization
- 1/2QV’s
- Stability of 1/2-QV’s in SrRuO
- 1/2 QV lattices
Triplet superfluidity, $d$-vector

A. Leggett, RMP 1975  Sigrist & Ueda RMP 1991
Triplet superfluidity, \textbf{d-vector}

A. Leggett, RMP 1975  Sigrist & Ueda RMP 1991

\[ \Delta_{s_3}(k) = - \sum_{k',s_3,s_4} V_{s's_3s_4}(k,k') \langle a_{k's_3} a_{-k's_4} \rangle \]
Triplet superfluidity, \textit{d}-vector

\begin{align*}
\Delta_{ss'}(k) &= -\sum_{k', s_3, s_4} V_{s's's_3's_4}(k, k') \langle a_{k's_3} a_{-k's_4} \rangle \\
\hat{\Delta}(k) &= -\hat{\Delta}^T(-k)
\end{align*}
Triplet superfluidity, **d-vector**

**Gap function**

\[ \Delta_{ss'}(k) = - \sum_{k', s_3, s_4} V_{ss's_3's_4}(k, k') \langle a_{k's_3} a_{-k's_4} \rangle \]

\[ \hat{\Delta}(k) = - \Delta^T(-k) \]

**Singlet gap function**

\[ \hat{\Delta}(k) = i \hat{\sigma}_y \psi(k) = \begin{pmatrix} 0 & \psi(k) \\ -\psi(k) & 0 \end{pmatrix} \]

A. Leggett, RMP 1975  
Sigrist & Ueda RMP 1991
Triplet superfluidity, \textit{d}-vector

A. Leggett, RMP 1975  Sigrist & Ueda RMP 1991

\begin{itemize}
  \item Gap function
    \[ \Delta_{ss'}(k) = - \sum_{k', s_3, s_4} V_{s's's_3's_4}(k, k') \langle a_{k's_3} a_{-k's_4} \rangle \]
    \[ \hat{\Delta}(k) = -\hat{\Delta}^T(-k) \]
  \item Singlet gap function
    \[ \hat{\Delta}(k) = i\hat{\sigma}_y \psi(k) = \begin{pmatrix} 0 & \psi(k) \\ -\psi(k) & 0 \end{pmatrix} \]
  \item Odd-Parity: \( \hat{\Delta}(k) \) is odd in \( k \), spin triplet
\end{itemize}
Triplet superfluidity, \textbf{d-}vector

**Gap function**

\[
\Delta_{ss'}(k) = - \sum_{k',s_3,s_4} V_{s'ss_3s_4}(k,k') \langle a_{k's_3} a_{-k's_4} \rangle
\]

\[
\hat{\Delta}(k) = -\hat{\Delta}^T(-k)
\]

**Singlet gap function**

\[
\hat{\Delta}(k) = i\hat{\sigma}_y \psi(k) = \begin{bmatrix} 0 & \psi(k) \\ -\psi(k) & 0 \end{bmatrix}
\]

**Odd-Parity:** \(\hat{\Delta}(k)\) is odd in \(k\), spin triplet

**Triplet gap matrix**

\[
\hat{\Delta}(k) = i(d(k) \cdot \hat{\sigma})\hat{\sigma}_y
\]

\[
= \begin{bmatrix}
-d_x(k) + id_y(k) & d_z(k) \\
-d_z(k) & d_x(k) + id_y(k)
\end{bmatrix}
\]

\[A. \text{ Leggett, RMP 1975} \quad \text{Sigrist & Ueda RMP 1991}\]
T-breaking (ABM)

\[ \Delta(k) = \Delta_0(T)(p_x \pm ip_y) \begin{pmatrix} -d_x + id_y & d_z \\ d_z & d_x + id_y \end{pmatrix} \]

where \( \hat{d} \) is a real unit vector

In plane \( \hat{d} \)

\[ d_z = 0 \text{ i.e., } d = (\cos \alpha, \sin \alpha, 0) \]

\[ \Delta(k) = \Delta_0(T)(p_x \pm ip_y) \begin{pmatrix} -e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]
$\frac{1}{2} QV \text{ with in-plane } \hat{d}$
1/2 QV with in-plane $\hat{a}$

The gap matrix

$$\Delta(k) = \Delta_0(T)(p_x \pm ip_y) \begin{pmatrix} -e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$
1/2 QV with in-plane \( \hat{d} \)

The gap matrix

\[
\Delta(\mathbf{k}) = \Delta_0(T)(p_x \pm ip_y) \begin{pmatrix} -e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}
\]

1/2 QV when \( \mathbf{d} = (\cos\alpha, \sin\alpha, 0) \)
1/2 QV with in-plane $\hat{d}$

The gap matrix

$$\Delta(\mathbf{k}) = \Delta_0(T)(p_x \pm ip_y) \begin{pmatrix} -e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

1/2 QV when $\mathbf{d} = (\cos\alpha, \sin\alpha, 0)$

$\iff 2\pi$ winding for only one spin component
1/2 QV with in-plane $\hat{d}$

- The gap matrix
  \[
  \Delta(\mathbf{k}) = \Delta_0(T)(p_x \pm ip_y) \begin{pmatrix} -e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}
  \]

- 1/2 QV when $\mathbf{d} = (\cos\alpha, \sin\alpha, 0)$
  - $2\pi$ winding for only one spin component
  - $\pi$ winding of order parameter phase $\phi$
  - $\pi$ rotation of $\mathbf{d}$ vector

$hc/2e$ vortex  $hc/4e$ vortices
Why? Exotic nature of $1/2$ QV in p+ip

Vortices of p+ip SF $\rightarrow$ zero modes at the core

Das Sarma, Tewari, Nayak (06)  Stone & Chung(06)  Ivanov(01)
Why? **Exotic** nature of 1/2 QV in p+ip

Vortices of p+ip SF $\rightarrow$ zero modes at the core

Kopnin and Salomaa PRB (1991)

Das Sarma, Tewari, Nayak (06) Stone & Chung(06) Ivanov(01)
Why? Exotic nature of 1/2 QV in p+ip

Vortices of p+ip SF → zero modes at the core
Kopnin and Salomaa PRB (1991)

Zero modes are Majorana

Das Sarma, Tewari, Nayak (06) Stone & Chung (06) Ivanov (01)
Why? Exotic nature of 1/2 QV in p+ip

Vortices of p+ip SF → zero modes at the core

Kopnin and Salomaa PRB (1991)

Zero modes are Majorana

- BdG qp’s
  \[ \gamma_i^\dagger = u\psi_i^\dagger + v\psi_i \]
  \[ \gamma_i^\dagger(E_m) = \gamma_i(-E_n) \]

- zero mode:
  \[ \gamma_i^\dagger(0) = \gamma_i(0) \]

Das Sarma, Tewari, Nayak (06) Stone & Chung (06) Ivanov (01)
Zero modes are Majorana

- Vortices of p+ip SF \rightarrow \text{zero modes at the core}
  Kopnin and Salomaa PRB (1991)

- Zero modes are 
  \[ \gamma_i^\dagger = u\psi_i^\dagger + v\psi_i \quad \gamma_i^\dagger(E_m) = \gamma_i(-E_n) \]
  \[ \gamma_i^\dagger(0) = \gamma_i(0) \]

- BdG qp’s
  Das Sarma, Tewari, Nayak (06) Stone & Chung(06) Ivanov(01)
Zero modes are **Majorana**

- **BdG qp’s** \( \gamma_i^\dagger = u\psi_i^\dagger + v\psi_i \), \( \gamma_i^\dagger(E_n) = \gamma_i(-E_n) \)
- **zero mode**: \( \gamma_i^\dagger(0) = \gamma_i(0) \)

**Majorana + vortex composite**

\[ \text{non-Abelian statistics} \]

Das Sarma, Tewari, Nayak (06), Stone & Chung (06), Ivanov (01)
Why? **Exotic** nature of 1/2 QV in p+ip

- **Vortices of p+ip SF** $\rightarrow$ **zero modes at the core**
  
  Kopnin and Salomaa PRB (1991)

- **Zero modes are Majorana**
  
  - BdG qp's $\gamma_i^\dagger = u\psi_i^\dagger + v\psi_i$ $\gamma_i^\dagger(E_m) = \gamma_i(-E_n)$
  
  - zero mode: $\gamma_i^\dagger(0) = \gamma_i(0)$

- **Majorana + vortex composite** $\rightarrow$ **non-Abelian statistics**

  Das Sarma, Tewari, Nayak (06) Stone & Chung(06) Ivanov(01)

**1/2 QV's: single Majorana zero mode**
5/2 state described as $p + ip$ paired stated of composite fermion

Pfaffian is real space many body BCS wave function of $p + ip$ SF

HQV is equivalent to $1/4$ qp

Moore & Read (91) Read & Green (00) Schriffer, p 48
In search of topological states with fractionalized excitations.

- Topological order and fractionalization
- 1/2QV’s
- **Stability of 1/2-QV’s in SrRuO**
- 1/2 QV lattices
Spin-triplet superconductivity in Sr$_2$RuO$_4$ identified by $^{17}$O Knight shift
Experiments?

- NMR on $^3$He-A thin films: $X$ Hakonen et al. Physica (89)
- Small angle neutron scattering: $X$ Riseman et al. Nature (98)
- Scanning SQUID imaging: $X$
  Dolocan et al, PRL (05), Bjorsson et al, PRB (05)
- NMR in the presence of $\mathbf{H} \perp ab$
  - $\mathbf{d} \parallel ab$: for $H \perp \approx 200$ G, Murakawa et al, PRL (04)
Energetics

Energy competition between full-QV and 1/2-QV
Energetics

- Energy competition between full-QV and 1/2-QV
  - Reducing vorticity saves magnetic energy
Energetics

- Energy competition between full-QV and 1/2-QV
  - Reducing vorticity saves magnetic energy
  - d-vector bending costs energy
Energetics

- Energy competition between full-QV and 1/2-QV
  - Reducing vorticity saves magnetic energy
  - d-vector bending costs energy

Gradient free energy when $d \perp L$ (London limit)

$$f_{\text{grad}}^{2D} = \frac{1}{2} \left( \frac{\hbar}{2m} \right)^2 \left[ \rho_s \left( \nabla_\perp \phi - \frac{2e}{\hbar c} A \right)^2 + \rho_{\text{sp}} (\nabla_\perp \alpha)^2 \right] + \frac{1}{8\pi} (\nabla \times A)^2$$
Energetics

- Energy competition between full-QV and 1/2-QV
  - Reducing vorticity saves magnetic energy
  - d-vector bending costs energy

Gradient free energy when $d \perp L$ (London limit)

$$f^2_{\text{grad}} = \frac{1}{2} \left( \frac{\hbar}{2m} \right)^2 \left[ \rho_s \left( \nabla_{\perp} \phi - \frac{2e}{\hbar c} A \right)^2 + \rho_{\text{sp}} \left( \nabla_{\perp} \alpha \right)^2 \right] + \frac{1}{8\pi} (\nabla \times A)^2$$

- Spin current energy diverges logarithmically!
Energetics

- Energy competition between **full-QV** and **1/2-QV**
  - Reducing vorticity **saves** magnetic energy
  - d-vector bending **costs** energy

- Gradient free energy when \( \mathbf{d} \perp \mathbf{L} \) (London limit)

\[
f_{\text{grad}}^{2D} = \frac{1}{2} \left( \frac{\hbar}{2m} \right)^2 \left[ \rho_s \left( \nabla_{\perp} \phi - \frac{2e}{\hbar c} \mathbf{A} \right)^2 + \rho_{sp} \left( \nabla_{\perp} \alpha \right)^2 \right] + \frac{1}{8\pi} \left( \nabla \times \mathbf{A} \right)^2
\]

- Spin current energy diverges logarithmically!

\[
\epsilon_{\text{sp}} = \frac{\pi}{4} \left( \frac{\hbar}{2m} \right)^2 \rho_{sp} \ln \left( \frac{R}{\xi} \right)
\]
Competition between screened magnetic repulsion and unscreened spin attraction

Finite equilibrium size for small $\rho_{sp}/\rho_s$

Leggett RMP 75
Mesoscopic sample

Sample of size ~ $\lambda$ a few micron

$L = 2\lambda$

Underway in Budakian lab

Sample of size ~ $\lambda$ a few micron
In search of topological states with fractionalized excitations.

- Topological order and fractionalization
- $1/2QV$'s
- Stability of $1/2$-$QV$'s in SrRuO
- $1/2$ QV lattices
1/2 QV Lattice?
1/2 QV Lattice?

Natural way to stabilize 1/2 QV
1/2 QV Lattice?

- Natural way to stabilize 1/2 QV
- Good track record for full QV lattice
  - Agterberg, PRB (98) predicted square lattice
  - T. Riseman et al., Nature (98) confirmed square lattice
1/2 QV Lattice?

- Natural way to stabilize 1/2 QV

- Good track record for full QV lattice
  - Agterberg, PRB (98) predicted square lattice
  - T. Riseman et al., Nature (98) confirmed square lattice

- Potential of tuning $\rho_{sp}/\rho_s$
  - Knowledge exist for $\rho_{sp}/\rho_s$ as a function of Fermi liquid parameters
  - p-wave Feshbach resonance
1/2 QV Lattice?

- SC (SF) with additional U(1) symmetry due to \( \hat{d} \) rotation

- Interlacing lattices of two types of vortices

- Different geometry depending on density and LL mixing

- Similar case arise in spinor condensate

\[ \Delta \alpha = \pm \pi \]
\[ \Delta \chi = \pi \]

Muller & Ho(02)
Barnett, Mukergee & Moore(08)
Prediction

- Minimize GL free energy to determine the VL structure
- Quartic terms in the free energy determine the structure

Field distribution as can be measured by neutron
Prediction

- Minimize GL free energy to determine the VL structure
- Quartic terms in the free energy determine the structure

Field distribution as can be measured by neutron
p-wave Feshbach resonance can allow for tuning for Fermi liquid parameters

\[
H = \sum_p \epsilon(p) a_p^\dagger a_p + \sum_{p,\alpha} \left[ \epsilon_\alpha + \frac{\epsilon(p)}{2} \right] b_{p\alpha}^\dagger b_{p\alpha} + \frac{1}{\sqrt{V}} \sum_{p,q,\alpha} g_{pq\alpha} \left( b_{q\alpha} a_{p+\frac{q}{2}}^\dagger a_{-p+\frac{q}{2}}^\dagger + \text{h.c.} \right)
\]

Gurarie, L. Radzihovsky, & A. V. Andreev (05)

Hope to arrive at a PD where \( \rho_{sp}/\rho_s \) can be tuned as a function of microscopic parameter
In search of topological states with half quantum vortices
In search of topological states with half quantum vortices
In search of topological states with half quantum vortices

1/2 QV's are not stable in bulk systems
In search of topological states with half quantum vortices

- 1/2 QV's are not stable in bulk systems
- Mesoscopic samples could favor 1/2 QV's
1/2 QV’s are not stable in bulk systems

Mesoscopic samples could favor 1/2 QV’s

1/2 QV Vortex Lattice can be pursued and detected