Chains of “Interacting” Non-Abelian Quasiparticles

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Work with: Huan Tran (FSU), Lukasz Fidkowski (Caltech), Kun Yang (FSU), Gil Refael (Caltech), Joel Moore (Berkeley).

L. Fidkowski, G. Refael, NEB, J. Moore arxiv: 0807.1123
H. Tran, NEB, in preparation.

Support: US DOE
Non-Abelian FQH States  (Moore, Read ’91)

Essential features:

A degenerate Hilbert space whose dimensionality is exponentially large in the number of quasiparticles.

States in this space can only be distinguished by global measurements provided quasiparticles are far apart.
SU(2)_k Non-Abelian Particles

1. Particles have topological charge \( s = 0, 1/2, 1, 3/2, \ldots, k/2 \)

   \[ \text{topological charge} = \frac{1}{2} \]

2. “Fusion Rule” for adding topological charge:

   \[
   s_1 \otimes s_2 = |s_1 - s_2| \oplus \left( |s_1 - s_2| + 1 \right) \oplus \cdots \oplus \min\left( s_1 + s_2, k - (s_1 + s_2) \right)
   \]

For example:

\[
\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1
\]

\( \rightarrow \)

Two \( \bullet \) particles can have total topological charge 0 or 1.

\( \alpha \begin{tikzpicture}
\draw[fill=green!80!black] (0,0) circle (0.3);
\draw[fill=green!80!black] (1,0) circle (0.3);
\node at (0.5,0) {0};\end{tikzpicture} \) + \( \beta \begin{tikzpicture}
\draw[fill=green!80!black] (0,0) circle (0.3);
\draw[fill=green!80!black] (1,0) circle (0.3);
\node at (0.5,0) {1};\end{tikzpicture} \)
$\mathcal{N} \to \infty$; Ordinary Spin-1/2 Particles

$\text{Dim}(N) \sim 2^N$
Non-crossing valence bond basis:

Any two particles connected by a bond form a singlet

Complete, linearly independent basis for the space of all singlet states.
Valence Bonds Basis

Nonorthogonal basis, but easy to compute with:

\[ |\alpha\rangle = \]

\[ |\beta\rangle = \]

\[ \langle \alpha | \beta \rangle = \]

\[ \langle \alpha | \beta \rangle = 2^{N_{\text{loops}} - N/2} = 2^{3-12/2} = 1/8 \]
$k = 4$

$\text{Dim}(N) \sim 3^{N/2}$
$k = 3$

$v = 12/5$ state?

$\dim(N) = \text{Fib}(N+1) \sim \phi^N$
2 = k

(k = 2

(v=5/2 state)

\[
\text{Dim}(N) = 2^{N/2-1}
\]
Quantum Dimension

Hilbert space of N particles with topological charge $\frac{1}{2}$ grows asymptotically as $d^N$ where $d$ is the “quantum dimension” of the particles.

\[ d = 2 \cos \frac{\pi}{k + 2} \]

<table>
<thead>
<tr>
<th>$k$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$\phi = \frac{1 + \sqrt{5}}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>2</td>
</tr>
</tbody>
</table>
Valence Bonds Basis for $SU(2)_k$

Non-crossing valence bond basis:

Any two particles connected by a bond fuse to trivial topological charge 0 if brought together.

A complete, but linearly dependent basis for the space of all states with total topological charge 0.
Valence Bonds Basis for SU(2)$_k$

Again, nonorthogonal, but still easy to compute with:

\[ |\alpha\rangle = \]

\[ |\beta\rangle = \]

\[ \langle\alpha|\beta\rangle = \]

\[ \langle\alpha|\beta\rangle = d^{N_{\text{loops}}-N/2} = d^{4-12/2} = 1/d^2 \]

\[ d = 2\cos\frac{\pi}{k+2} \]

Quantum Dimension
Interacting Non-Abelian Anyons

Localized quasiparticles

Topological degeneracy is lifted when quasiparticles are close together (for FQHE states, this means within a few magnetic lengths).

Assume trivial topological charge is energetically favored:

\[ H = - \sum_{i,j} J_{i,j} \prod_{i,j}^{0} ; \quad J_{i,j} > 0 \]

Projection onto state of particle \( i \) and \( j \) with total topological charge 0.
Assume trivial topological charge is energetically favored:

$$H = - \sum_i J_i \Pi_i^0 ; \quad J_i > 0$$

( $\Pi_i^0 \equiv \Pi_{i,i+1}^0$ )
Uniform $SU(2)_k$ Chains

$k \to \infty$  Ordinary spin-1/2 AFM Heisenberg model:

$$H = -J \sum_i \prod_i^0 = -J \sum_i \left( \frac{1}{4} - S_i \cdot S_{i+1} \right)$$

Conformally invariant quantum critical model with central charge: $c=1$

Uniform $SU(2)_k$ chains can be mapped onto exactly solvable Andrews-Baxter-Forrester models which realize minimal CFTs with central charges,

$$c = 1 - \frac{6}{(k+1)(k+2)}$$  

(Feiguin et al., PRL 98, 160409 (2007).)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$c$</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k \to \infty$</td>
<td>$c=1$</td>
<td>Heisenberg Model</td>
</tr>
<tr>
<td>$k=3$</td>
<td>$c=\frac{7}{10}$</td>
<td>Golden Chain</td>
</tr>
<tr>
<td>$k=2$</td>
<td>$c=\frac{1}{2}$</td>
<td>Critical TFIM</td>
</tr>
</tbody>
</table>
Random SU(2)_k Chains

Given the similarity between ordinary spin and SU(2)_k particles we can apply the real space RG. (Ma, DasGupta, Hu ‘79, D. Fisher ’94)
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Random $SU(2)_k$ Chains

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$$H = - \sum_i J_i \Pi_i^0$$

**Diagram:**

- $J_i$ axis
- Bonds $J_1$, $J_2$, $J_3$
- Strongest Bond
Random SU(2)$_k$ Chains

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$H = - \sum_i J_i \Pi^0_i$
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\[ H = - \sum_i J_i \Pi_i^0 \]

Effective interaction from 2$^{nd}$ order perturbation theory

\[ \tilde{J} = \frac{1}{2} \frac{J_1 J_3}{J_2} \]

Spin-1/2 particles
(Ma, DasGupta, Hu ’79)

Strongest Bond
Random $SU(2)_k$ Chains

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$$H = -\sum_i J_i \Pi_i^0$$

Effective interaction from $2^{nd}$ order perturbation theory

$$\tilde{J} = \frac{2 J_1 J_3}{d^2 J_2}$$

$SU(2)_k$ particles (NEB, K.Yang, PRL’07)
Random SU(2)_k Chains

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Random Singlet Phase for SU(2)_k particles: Bonds freeze into a particular non-crossing valence-bond state.

(NEB, K. Yang, PRL ’07)
Random SU(2)_k Chains

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Infinite Random Fixed Point (D. Fisher ‘94)

\[ L^{1/2} \sim \ln \frac{1}{E} \quad \text{Specific Heat:} \quad C \propto \frac{1}{|\ln T|^3} \]
Random SU$(2)_k$ Chains

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$H = - \sum_i J_i \Pi_i^0$

Specific Heat:

$(L. Fidkowski, G. Refael, NEB, J. Moore, arxiv:0807.1123)$
Entanglement Entropy

A quantum system composed of two parts: A and B

\[ \rho_A = \text{Tr}_B \left[ |GS\rangle \langle GS| \right] \quad \rightarrow \quad S_A \equiv -\text{Tr} \left[ \rho_A \log_2 \rho_A \right] \]

Reduced density matrix  Entanglement entropy

Simple example: An SU(2) singlet bond

\[ \rho_A = \frac{1}{\sqrt{2}} \left( \uparrow_A \downarrow_B - \downarrow_A \uparrow_B \right) \]

\[ \rho_A = \frac{1}{2} |\uparrow_A \rangle \langle \uparrow_A | + \frac{1}{2} |\downarrow_A \rangle \langle \downarrow_A | \quad \rightarrow \quad S_A = 1 \]
Entanglement Entropy

At 1+1 dimensional conformally invariant quantum critical points, the entanglement entropy scales logarithmically with the size of region A with a universal coefficient:

\[ S(L) \approx \frac{c}{3} \log_2 L \]

\[ c = \text{central charge} \]

(Holzhey et al. ‘94, Calabrese & Cardy ‘04)

For uniform Heisenberg model (c=1)

For uniform critical TFIM (c=1/2)
Entanglement Entropy of Random Spin-1/2 Chains
(Refael & Moore PRL 93, 260602 (2004))

In the random singlet phase the entanglement entropy also scales logarithmically with $L$

$$S(L) \approx (\text{entropy per bond}) \times \frac{\ln 2}{3} \log_2 L \approx \frac{\ln 2}{3} \log_2 L$$

Avg. # of bonds leaving region of length $L \approx \frac{1}{3} \ln L = \frac{\ln 2}{3} \log_2 L$

"effective" central charge: $\tilde{c} = \ln 2$
For $SU(2)_k$ random chains the only thing that is different is the entanglement per bond.

Imagine $N \gg 1$ “singlet” pairs:

$$S_A \approx \log_2 d^N = N \log_2 d$$

Entropy per bond $= \log_2 d$
In the random singlet phase the entanglement entropy also scales logarithmically with $L$

\[ \text{Avg. \# of bonds leaving region of length } L \approx \frac{1}{3} \ln L = \frac{\ln 2}{3} \log_2 L \]

\[ S(L) \approx (\text{entropy per bond}) \times \frac{\ln 2}{3} \log_2 L \approx \frac{\ln d}{3} \log_2 L \]

“effective” central charge: $\bar{c} = \ln d$
Valence-Bond Monte Carlo
(Sandvik, PRL 95, 207203 (2005))

Idea: Project out ground state of $H$ by repeatedly applying $-H$ to some initial valence-bond state $|S_0\rangle$

\[
(-H)^n|S_0\rangle = \sum_{i_1\cdots i_n} J_{i_1} \cdots J_{i_n} \prod_{i_1}^0 \cdots \prod_{i_n}^0 |S_0\rangle = \sum_\alpha w(\alpha) |\alpha\rangle
\]

Initial valence-bond state

Sum over “non-crossing” valence-bond states.

Weight factors $w(\alpha)$ are easy to compute and update for efficient Monte Carlo sampling. Straightforward to generalize to SU(2)$_k$ particles.
Valence-Bond Entanglement
(Alet, Capponi, Laflorencie, Matthieu, PRL 99, 117204 (2007))

For the ground state wavefunction \(|GS\rangle = \sum w(\alpha) |\alpha\rangle\)
the “valence-bond entanglement” is defined to be:

\[ S_{VB}(L) = \sum_{\alpha} \frac{w(\alpha) S(L;\alpha)}{\sum_{\alpha} w(\alpha)} \]

Entanglement entropy in the valence-bond state \(|\alpha\rangle\) computed a la Refael and Moore.

Exact result for uniform chains (Jacobsen & Saleur, PRL 100, 087205 (2008))

\[ S_{VB}(L) \approx \frac{4 \ln d}{\pi^2} \frac{1}{k+1} \frac{d}{\sqrt{4-d^2}} \log_2 L \]

Close to, but not exactly equal to c/3
Valence-Bond Entanglement: Uniform Case

$k \to \infty$ case first studied by Alet et al. ‘07

H. Tran, NEB, in preparation
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If bonds “freeze” on long length scales then $S_{VB}(L)$ should show the same scaling as the “true” entanglement $S(L)$ for large $L$.

→ For random chains expect:

$$S_{VB}(L) \approx \frac{\ln d}{3} \log_2 L$$
For random chains, how do we know bonds are “freezing”? Look at fluctuations in number of bonds leaving region of size $L$. 
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\[ L \]
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![Diagram showing fluctuations in number of bonds leaving a region of size $L$.]
For random chains, how do we know bonds are “freezing”? 

Look at fluctuations in number of bonds leaving region of size $L$. 

![Diagram showing fluctuations in bond number](image)
For random chains, how do we know bonds are “freezing”? Look at fluctuations in number of bonds leaving region of size $L$.

If bonds are frozen, only fluctuations near boundary of region change the number of bonds leaving that region.

$\sigma_n^2 = \left\langle \left\langle n_L^2 \right\rangle - \left\langle n_L \right\rangle^2 \right\rangle$

Average over disorder

Bond fluctuations for particular realization of disorder

Expect $\sigma_n^2$ to be independent of $L$ for large $L$ if bonds freeze.
Bond Fluctuations: Signature of Freezing

\[ \langle S_L^2 \rangle - \langle S_L \rangle^2 \]

Block size $L$

H. Tran, NEB, in preparation
Bond Fluctuations: Signature of Freezing

\[ \langle \langle S_L^2 \rangle - \langle S_L \rangle^2 \rangle \]

Exact result
(Jacobsen & Saleur, PRL ’08)

Uniform chain

H. Tran, NEB, in preparation
Bond Fluctuations: Signature of Freezing

\[ \langle (S_L^2 - \langle S_L \rangle) \rangle \]

- Uniform chain
- Random chain

Exact result
(Jacobsen & Saleur, PRL ’08)

H. Tran, NEB, in preparation
Valence-Bond Entanglement: Random Case

$k \rightarrow \infty$ case first studied by Alet et al. PRL ‘07

H. Tran, NEB, in preparation
Valence-Bond Entanglement: Random Case

\[ S_L = \frac{\ln d}{3} \log_2 L + \text{const.} \]

\[ k \to \infty \]

\[ k = 3 \]

\[ k = 2 \]

Block size \( L \)

H. Tran, NEB, in preparation

The \( k \to \infty \) case was first studied by Alet et al. PRL ‘07.
Conclusions

There is a close analogy between the properties of SU(2)_k non-Abelian quasiparticles and ordinary spin-1/2 particles.

Chains of interacting non-Abelian particles can enter “random singlet phases,” analogous to those arising in random spin-1/2 chains.

Universal entanglement scaling.

$$S(L) \approx \frac{\ln d}{3} \log_2 L$$