Effective Field Theories of Topological Insulators

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AtMa Chan, Shinsei Ryu, Taylor Hughes and EF, ArXiv:1210.4305

Motivation

- To provide an effective field theory of topological phases
- We know how to do this for the fractional quantum Hall states (Wen)
- Hydrodynamic low energy theory for the charge degrees of freedom
- Distinction between topologically protected response and topological phase: when is a topological phase?
- Candidates: in 2D Chern-Simons; in 3D: "axion"; "BF" theory
- How are topological insulators related to topological phases?
- 2D vs 3D; U(1) vs Z_2
- Fractionalization

Strategy

- We will use "functional bosonization" to derive the effective field theory
- It is equivalent to bosonization in D=1+1 as an operator identity for **free gapless fermions**
- For D>1+1 we will get an effective hydrodynamic field theory for **gapped fermions**
- Extension to interacting fermions and fractionalization

Functional Bosonization

Bosonization of the fermion path integral

$$Z[A^{\rm ex}] = \int \mathcal{D}\left[\bar{\psi}, \psi\right] \exp\left(iS_F[\bar{\psi}, \psi, A^{\rm ex}]\right)$$

To compute current correlators

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2) \cdots \rangle = \frac{1}{i} \frac{\delta}{\delta A^{\rm ex}_{\mu_1}(x_1)} \frac{1}{i} \frac{\delta}{\delta A^{\rm ex}_{\mu_2}(x_2)} \cdots \ln Z[A^{\rm ex}]$$

Use gauge invariance of the fermion path integral: shift A^{ex} to $A^{ex} + a$, where a is a gauge transformation: $f_{\mu\nu}[a] = 0$

$$Z[A^{\rm ex} + a] = Z[A^{\rm ex}].$$

$$Z[A^{\rm ex}] = \int \mathcal{D}[a]_{\rm pure} Z[A^{\rm ex} + a]$$

$$Z[A^{\rm ex}] = \int \mathcal{D}[a,b] Z[a] \times \exp\Big(-\frac{i}{2} \int d^D x\, b_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} \left(f_{\alpha\beta}[a] - f_{\alpha\beta}[A^{\rm ex}]\right) \Big)$$

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2) \cdots \rangle = \langle \epsilon^{\mu_1\nu_1\lambda_1\cdots} \partial_{\nu_1} b_{\lambda_1\dots}(x_1) \epsilon^{\mu_2\nu_2\lambda_2\cdots} \partial_{\nu_2} b_{\lambda_2\dots}(x_2) \cdots \rangle$$

$$j^\mu(x) \quad \equiv \quad \epsilon^{\mu\nu\lambda\rho\cdots} \partial_\nu b_{\lambda\rho\dots}(x) \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

- This procedure is meaningful only if the effective action is local
- This works in 1+1 dimensions for gapless relativistic fermions
- For D>1+1 it works only if there is a finite energy gap
- This leads to a hydrodynamic description
- For systems with a Fermi surface one obtains the Landau theory of the Fermi liquid

Example: Polyacetylene

- Fermions in d=1 with a spontaneously broken translation symmetry: broken chiral symmetry (Class AIII)

$$\begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix} \to e^{i\sigma_3\theta}\begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix} \qquad \rho(x)\to\rho(x+a) \Rightarrow \theta = k_F a$$

Topological invariant
$$\nu = \frac{\theta(+\infty)-\theta(-\infty)}{2\pi}$$

Charge conjugation (particle-hole)
$$\theta = \nu\pi \mod 2\pi$$

$$Z[A^{\text{ex}}] = \int \mathcal{D}[a,b]\exp\left(i\int d^D x\mathcal{L}\right)$$

$$\mathcal{L} = -b\epsilon^{\mu\nu}\partial_\mu(a_\nu - A^{\text{ex}}_\nu) + \frac{\theta}{2\pi}\epsilon^{\mu\nu}\partial_\mu a_\nu + \cdots$$

D=2+1 Chern Insulator (Class A or D)

- Free fermions with broken time reversal invariance: integer quantum Hall states and the quantum anomalous Hall state
- The states are characterized by a topological invariant, the Chern number *Ch*
- In this case we obtain

$$\mathcal{L} = -b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda^{\rm ex}) + \frac{\mathsf{Ch}}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda.$$

- The first term is the *BF* Lagrangian
- The hydrodynamic field *b* couples to flux tubes
- The statistical gauge field *a* coupes to quasiparticle worldlines

Quantized Hall conductance $\sigma_{xy} = Ch\ \frac{e^2}{h}$

3D Topological Insulator (Class AIII and DIII)

- Example: massive relativistic fermions with a conserved U(1) charge.
- This system has a topological invariant: the winding number

$$\mathcal{L} = -b_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda(a_\rho - A^{\text{ex}}_\rho) + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu \partial_\lambda a_\rho - \frac{1}{4\pi^2 g^2}\partial_\mu a_\nu \partial^\mu a^\nu + \cdots$$

If time-reversal (particle-hole) is imposed, the topological class is Z_2

$$\text{with } \theta = \nu\pi \mod 2\pi$$

The effective action for the external gauge field has an axion term (Qi, Hughes, Zhang, 2009)

D=4+1 dimensional Topological Insulator

In D=4+1dimensions the effective field theory is a *BF* theory with a topological term for the field *a* whose coefficient is a topological invariant, the second Chern number Ch_2

$$\mathcal{L} = -b_{\mu\nu\lambda}\epsilon^{\mu\nu\lambda\rho\sigma}\partial_\rho(a_\sigma - A^{\text{ex}}_\sigma) + \frac{\mathsf{Ch}_2}{24\pi^2}\epsilon^{\mu\nu\lambda\rho\sigma}a_\mu\partial_\nu a_\lambda\partial_\rho a_\sigma + \cdots$$

This theory can be mapped onto the result of Cho and Moore:

$$\mathcal{L} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}a_\mu\partial_\nu b_{\lambda\rho} + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}A^{\text{ex}}_\mu\partial_\nu b_{\lambda\rho} + C\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu\partial_\lambda A^{\text{ex}}_\rho$$

- Conclusion: all free fermion insulators (or adiabatically equivalent states) have a hydrodynamic effective field theory with the form of a *BF* action with or without a topological term.
- In general the hydrodynamic field is a tensor (antisymmetric) field
- The structure of the topological term depends on the dimension
- For systems on closed manifolds this approach involves an average over boundary conditions

Physical Picture

The effective field theory we found (without the topological term) has the same form as the topological field theory of a d=2 gapped superconductor (Hansson, Oganesyan and Sondhi)

$$\mathcal{L} = -\frac{2k}{4\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\sigma - A_\lambda^{\text{ex}}) + \cdots$$

- Whereas in our case k=1, in the SC k=2.
- for k>1 this is a fractionalized state
- In the case of the SC fractionalization follows from pairing: the Bogoliubov quasiparticles have spin but no charge

Topological Insulator in 2+1 Dimensions (Class A or D)

$$\mathcal{L} = -\frac{2k}{4\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda^{\text{ex}}) + \frac{Ch}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$
$$= \frac{K_{ij}}{4\pi} \alpha_\mu^i \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_\lambda^j + \frac{k}{2\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda^{\text{ex}}$$

$$K = \begin{pmatrix} 0 & -k \\ -k & Ch \end{pmatrix} \qquad (\alpha_\mu^1, \alpha_\mu^2) = (b_\mu, a_\mu)$$

- In general the effective field theory has the *K*-matrix form (as in the fractional quantum Hall fluids) (Wen and Zee) with a degeneracy k^2 on the torus
- A theory with k>1 is a fractionalized topological Chern insulator
- To derive this state once has to resort to flux attachment or parton constructions

Electromagnetic Response

The effective action for the external gauge field is

$$S_{\text{eff}}[A^{\text{ex}}] = \frac{1}{8\pi^2} \int d^4x\, \theta \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu^{\text{ex}} \partial_\rho A_\sigma^{\text{ex}}$$

If the bulk mass gap is allowed to have a chiral twist, one finds a current

$$j^\mu = \epsilon^{\mu\nu\lambda\rho} \partial_\nu b_{\lambda\rho} = \frac{1}{4\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\nu \left(\theta \partial_\lambda A_\rho^{\text{ex}}\right)$$

If the chiral angle has a jump from 0 to π at the surface one obtains a surface Hall conductivity with 1/2 of the quantized value

$$\sigma_{xy} = \frac{1}{4\pi} = \frac{e^2}{2h}$$

Fractionalized Z_2 Topological Insulator

- The fractionalized U(1) topological insulator in 2+1 is effectively equivalent of a FQH state
- Not much is known on a D=3+1 fractionalized Z_2 insulator
- One can obtain an effective field theory using a parton construction (flux attachment does not work)
- In the simplest case we get

$$\mathcal{L} = -\epsilon^{\mu\nu\lambda\rho} b_{\mu\nu} \partial_\lambda \sum_{I=1}^{k} a_\rho^I + \frac{\theta}{8\pi^2} \sum_{I=1}^{k} \epsilon^{\mu\nu\lambda\rho} \partial_\mu a_\nu^I \partial_\lambda a_\rho^I - e\epsilon^{\mu\nu\lambda\rho} \partial_\nu b_{\lambda\rho} A_\mu^{\text{ex}}$$

Projection: $a_\mu^I \equiv a_\mu \quad \Rightarrow \quad \mathcal{L} = \frac{\theta}{8\pi^2 k} \epsilon^{\mu\nu\lambda\rho} \partial_\mu A_\nu^{\text{ex}} \partial_\lambda A_\rho^{\text{ex}}$