Topological Quantum Field Theory of Symmetry-Protected Topological Phases

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ICMT seminar
October 5th, 2015
One-to-one Correspondence between Group Cohomology and TQFT

\[ \mathcal{H}^4(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots , U(1)) = \prod_{I<J}(\mathbb{Z}_{N_{IJ}})^2 \times \prod_{I<J<K}(\mathbb{Z}_{N_{IJK}})^2 \times \prod_{I<J<K<L}\mathbb{Z}_{N_{IJKL}} \]

<table>
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<th>Symmetry</th>
<th>Topological Quantum Field Theory</th>
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Topological Phases of Matter
Everyone is talking about topological.

- Topological phases: infinite energy gap, no finite energy scale. Perturbation correction of any order is irrelevant.
- Only thing matters: equivalence class
- e.g, UIUC is topologically equivalent to Tsinghua University in Beijing.
Charles Lane and Tsin Chuang (庄俊, 1909-1914 studied Agricultural Engineering at UIUC).

constructed during 1917-1920
Grand Auditorium at Tsinghua University
清华大学大礼堂
Intrinsic Topological phases of matter

- Boundary has `intrinsic’ anomaly (under no additional conditions!)
- Spectrum gap. *(defined by a proper sequence of Hamiltonians)*
- The fixed point partition function is a **TQFT**. *(Topological QFT)*
- TQFT action=wedge product of differential forms (diffeomorphism invariant).
Chern-Simons Theory of Abelian Anyons

Abelian Chern-Simons theory (Wen, Zee, Frohlich, etc.) describes Abelian anyon systems

\[ S = \int \left( \frac{iK^{IJ}}{4\pi} a^I \wedge da^J + l_I a^I \wedge \ast j + \frac{q^I}{2\pi} A \wedge da^I \right) \]
Boundary Anomaly

For simplest 1/3 Laughlin State: Perturbative grav. anomaly.

\[ K = 3 ; l = 1, 2, 3 ; q = 1 \]

\[ S = \int \left( \frac{i K^{I} J}{4\pi} a^{I} \wedge da^{J} + l_{I} a^{I} \wedge \ast j + \frac{q^{I}}{2\pi} A \wedge da^{I} \right) \]

If we try to define edge as a standard open chain:

perpetual motion machine, mysterious Power station

Energy disappears
Symmetry-Protected Topological Phases 

(Almost trivial states)

- Non-intrinsic topological phases of matter.
- Bulk is gapped and non-fractionalized.
- Distinct SPT phases in symmetry-preserving parameter space

Hamiltonian: $H(g_1, g_2, g_3)$
Groundstate: $\Phi(g_1, g_2, g_3)$

$g_3 < 0$
Symmetry-Protected Topological Phases

*Almost trivial states*

- **Non-intrinsic** topological phases of matter.
- Bulk is gapped and non-fractionalized.
- One phase in symmetry-breaking parameter space

Hamiltonian: $H(g_1, g_2, g_3)$

Groundstate: $\Phi(g_1, g_2, g_3)$
Topological Insulators as a *Fermionic SPT*

\[ U(1) \times \text{TRS} \]

- In symmetry preserving parameter space, two phases.
- Surface anomaly: Odd # of Dirac cones (violates 2D fermion-doubling theorem)

Hamiltonian: \( \mathcal{H}(g_1, g_2, g_3) \)
Groundstate: \( \Phi(g_1, g_2, g_3) \)

\[ g_3 < 0 \]
Topological Insulators as a Fermionic SPT

$U(1) \rtimes \text{TRS}$

In symmetry breaking parameter space, one phase.

# of Dirac cones are not restricted.

Hamiltonian: $H(g_1, g_2, g_3)$
Groundstate: $\Phi(g_1, g_2, g_3)$
SPT in 1D
1D SPT Phase (Haldane Phase) with SO(3) Symmetry

Bulk is Nonlinear Sigma Model with Topological Theta Term: $g \to \infty$

$$S = \int dx d\tau \left( \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \frac{i\Theta}{8\pi} \epsilon^{\mu\nu} \mathbf{n} \cdot \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n} \right)$$

$$\Theta = 2\pi \mod 4\pi$$

T. K. Ng 1994; Cenke Xu, etc.
Z₂ Classification of Haldane Phase

\[ S = \int dx d\tau \sum_{I}^2 \left( \frac{1}{g} (\partial_\mu n^I)^2 + \frac{i\Theta}{8\pi} \epsilon^{\mu\nu} n^I \cdot \partial_\mu n^I \times \partial_\nu n^I \right) + \lambda \int dx d\tau n^1 \cdot n^2 \]

- Couple two Haldane chains via SO(3) invariant interactions.

- Bulk gap is not closed.

- Theta Periodicity is 4\pi

- S=6 \( \Theta = 12\pi \)
- S=5 \( \Theta = 10\pi \)
- S=4 \( \Theta = 8\pi \)
- S=3 \( \Theta = 6\pi \)
- S=2 \( \Theta = 4\pi \)
- S=1 \( \Theta = 2\pi \)
SPT in 2D
2D SPT phases with Abelian Symmetry

- ‘Almost trivial’ Chern-Simons theory.
- Classify distinct symmetry transformations on Chern-Simons theory on a 2D disk. (Lu, Vishwanath 2012)

\[ S = \frac{i}{4\pi} \int K^{IJ} a^I \wedge da^J \]

\[ K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
2D U(1) SPT: Integer Quantum Hall Effect of Bosons

`Almost trivial’ Chern-Simons theory with exotic symmetry twist

\[ \mathcal{L} = \frac{i}{4\pi} \left( a^1_\mu a^2_\mu \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_\nu \begin{pmatrix} a^1_\lambda \\ a^2_\lambda \end{pmatrix} \epsilon^{\mu\nu\lambda} + \frac{i}{2\pi} A_\mu (1 \quad n) \begin{pmatrix} \partial_\nu a^1_\lambda \\ \partial_\nu a^2_\lambda \end{pmatrix} \epsilon^{\mu\nu\lambda} \]

Hall conductance:

\[ S[A] = \frac{\sigma_{xy}}{4\pi} \int A \wedge dA \]

\[ \sigma_{xy} = \frac{1}{2\pi} (1 \quad n) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ n \end{pmatrix} = \frac{1}{2\pi} \times 2n \]

\( n \in \mathbb{Z} \)
Exotic $2\pi$ Vortex Condensation: Physical understanding of U(1) SPT order

- $2\pi$ vortex condensation (no topological order)
  \[ L = (\partial_\mu \theta - a_\mu - nA_\mu)^2 + i1A_\mu J_\mu \]

- $2\pi$ vortex also carries $n$ unit of bare electric charge.

- Higgs mechanism leads to:
  \[ a_\mu + nA_\mu = 0 \]

\[ S[A] = \frac{\sigma_{xy}}{4\pi} \int A \wedge dA \quad \sigma_{xy} = \frac{1}{2\pi} \times 2n \]
Start with a free fermion Theory:

\[ \mathcal{L} = \frac{1}{4\pi} \left( a_\mu^1 a_\mu^0 a_\mu^{-1} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \partial_\nu \left( \begin{array}{c} a_\lambda^1 \\ a_\lambda^0 \\ a_\lambda^{-1} \end{array} \right) \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} A_\mu \partial_\nu \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \left( \begin{array}{c} a_\lambda^1 \\ a_\lambda^0 \\ a_\lambda^{-1} \end{array} \right) \epsilon^{\mu\nu\lambda} \]

Impose projection, \( \sum d a^I = 0 \), single occupancy)

end up with interacting boson theory:

\[ \mathcal{L} = \frac{1}{4\pi} \left( a_\mu^1 a_\mu^{-1} \right) \left( \begin{array}{ccc} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \partial_\nu \left( \begin{array}{c} a_\lambda^1 \\ a_\lambda^{-1} \end{array} \right) \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} A_\mu \partial_\nu \left( \begin{array}{c} 1 \\ -1 \\ 1 \end{array} \right) \left( \begin{array}{c} a_\lambda^1 \\ a_\lambda^{-1} \end{array} \right) \epsilon^{\mu\nu\lambda} \]

\( \sigma = 2 - 0 = 2 \)

Trivial insulators

Nontrivial U(1) SPT: bosonic quantum Hall insulator

PY, Wen, PRB (2013); Lu, Lee PRB (2014); Liu, Mei, PY, Wen PRB (2014)
Comparison: boundary anomaly of 2D U(1) SPT and 2D Chiral Topo. Order

If we try to define edge as a standard open chain:

U(1) SPT:

Laughlin state:

mysterious Charge source  Charge disappears
perpetual motion machine, mysterious Power station  Energy disappears
SPT in 3D
Difficulty and Exoticity in 3D

- point-particle excitations, loop excitations.
- 1-form, 2-form gauge fields.
- More topological terms (only $ada$, $a^1a^2a^3$ in 2D).
- 2D boundary can be topologically ordered (Senthil, Vishwanath 2012).
Questions to be addressed:

- Q: How to *physically* derive TQFT?
- Q: How to impose global symmetry?
- Q: How to determine the level quantization?
Domain Wall Condensation (DWC)

- $Z_{N_I}$ rotors + $Z_{N_I}$ gauge fields
- **Turn off** $Z_{N_I}$ gauge fields:
  Spontaneous Symmetry Breaking.
- Domain walls are finite energy topo. excitations.
- **Turn on** gauge fields:
  Domain Walls Condense (DWC) $\rightarrow$ symmetry restoration.

\[
S = \frac{1}{2} \sum_i \sum_{\langle ij \rangle} \left( \frac{\theta_i^{I}}{N_I} - \frac{\theta_j^{I}}{N_I} - \frac{a_{ij}^{I}}{N_I} - 2\pi \frac{L_{ij}^{I}}{N_I} \right)^2
\]

\[
\theta_i^{I} = \frac{2\pi}{N_I} k_i^{I}, k_i^{I} \in Z_{N_I}, a_{ij}^{I} = \frac{2\pi}{N_I} K_{ij}^{I},
\]

\[
K_{ij}^{I} = -K_{ji}^{I} \in Z_{N_I}, L_{ij}^{I} = -L_{ji}^{I} \in Z
\]

$D(Z_{N_2})$
Long-wavelength theory of trivial DWC

- Duality Transformation leads to BF term at level-1:

\[ S = \sum I \int \frac{i}{2\pi} b^I \wedge d\alpha^I \]

- Sym. implementation via Modifying compactification conditions:

\[
\frac{1}{2\pi} \int_{\mathcal{M}^3} db^I \rightarrow \frac{1}{2\pi} \int_{\mathcal{M}^3} db^I + N_I ,
\]

symmetry charge is conserved mod \(N_I\)
DWC in the presence of Contact Topological Term

- Our working assumption: Trivial DWC always leads to trivial SPT.
- Pure BF term at level-1 = always trivial SPT.
- Add a `Contact Topological Term’:

\[
S = \frac{1}{2} \sum_i \left[ (\Delta_\mu \theta^1_i - a^1_\mu)^2 + (\Delta_\mu \theta^2_i - a^2_\mu)^2 + (\hat{\Delta}_\mu \theta^3_i - a^3_\mu)^2 \right] \\
- i\rho \sum_i \left[ (\Delta_\mu \theta^1_i - a^1_\mu)(\Delta_\nu \theta^2_i - a^2_\nu)(\hat{\Delta}_\lambda a^3_\rho) \epsilon^{\mu\nu\lambda\rho} \right],
\]
Physical Picture of Contact Term:
Bosonic wormhole effect

\[ S = \frac{1}{2} \sum_i \left[ (\Delta_\mu \theta^1_i - a^1_\mu)^2 + (\Delta_\mu \theta^2_i - a^2_\mu)^2 + (\Delta_\mu \theta^3_i - a^3_\mu)^2 \right] \]

\[-i \rho \sum_i \left[ (\Delta_\mu \theta^1_i - a^1_\mu)(\Delta_\nu \theta^2_i - a^2_\nu)(\hat{\Delta}_\lambda a^3_\rho)\epsilon^{\mu\nu\lambda\rho} \right],\]
Long-wavelength theory (TQFT) of Exotic DWC:

- Duality Transformation leads to:
  \[ S_{\text{TQFT}} = \frac{i}{2\pi} \sum_{I}^{3} \int b^{I} \wedge d a^{I} + i p \int a^{1} \wedge a^{2} \wedge d a^{3} + S_{M} , \]

- Sym. implementation via Modifying compactification conditions:
  \[ \frac{1}{2\pi} \int_{\mathcal{M}^{3}} db^{I} \rightarrow \frac{1}{2\pi} \int_{\mathcal{M}^{3}} db^{I} + N_{I} , \]

Defining Properties of \( Z_{N_{I}} \) symmetry charge
Gauge Transformations and Wilson Loop / Surface / Volume operators

\[
S_{\text{TQFT}} = \frac{i}{2\pi} \sum_{I}^{3} \int b^{I} \wedge da^{I} + ip \int a^{1} \wedge a^{2} \wedge da^{3} + S_{M},
\]

- Gauge transformations:

\[
a^{I} \rightarrow a^{I} + d\chi^{I}, \quad b^{I} \rightarrow b^{I} + dV^{I} - 2\pi p \epsilon^{IJ3} \chi^{J} da^{3},
\]

\[
\exp\{i \int_{\mathcal{M}^{1}} a^{I}\} \quad \exp\{i \int_{\mathcal{M}^{2}} b^{I} - i2\pi p \int_{\mathcal{V}^{3}} \epsilon^{IJ3} a^{J} \wedge da^{3}\}
\]

\[
\partial \mathcal{V}^{3} = \mathcal{M}^{2}.
\]
Level Quantization Required by Consistent Implementation of Global Symmetry

\[ S_{TQFT} = \frac{i}{2\pi} \sum_I^3 \int b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge da^3 + S_M, \]

\[ p = \frac{k N_1 N_2}{4\pi^2 N_{12}}, \quad k \in \mathbb{Z}_{N_{123}} \]

- Under gauge transformation of \( b^I_{\mu\nu} \)
  \[ b^I \rightarrow b^I + dV^I - 2\pi p \epsilon^{IJ3} \chi^J \, da^3 \]
  Symmetry charge is changed:
  \[ \frac{1}{2\pi} \int_M \, db^I \rightarrow \frac{1}{2\pi} \int_M \, db^I - p \int_M \, d\chi^J \wedge da^3 \epsilon^{IJ3} \]

- The additional charge has to be mod \( N_I \),
  (\( \mathbb{Z}_{N_I} \) symmetry charge is conserved mod \( N_I \))
Level Quantization Required by Consistent Implementation of Global Symmetry

\[2\pi p \int_{M^3} d\chi^1 \wedge da^3 = 2\pi N_2 \times \mathbb{Z}\]
\[2\pi p \int_{M^3} d\chi^2 \wedge da^3 = 2\pi N_1 \times \mathbb{Z}\]

\[p = \frac{k}{4\pi^2} \frac{N_1 N_2}{N_{12}}, \quad k \in \mathbb{Z}\]
**Level Identification due to Redundancy under Level Shifting Operation**

- **Shifting operation:** two levels might label the same ground state.

\[
S_{\text{TQFT}} = \frac{i}{2\pi} \sum_{I} \int b^I \wedge d\alpha^I + ip \int a^1 \wedge a^2 \wedge d\alpha^3 + S_M,
\]

- \( b^3 \rightarrow b^3 - \frac{K^3 N_1 N_2}{2\pi N_{12}} a^1 \wedge a^2 \),

- \( b^1 \rightarrow b^1 + \frac{K^1 N_1 N_2}{2\pi N_{12}} a^2 \wedge a^3 \),

- \( b^2 \rightarrow b^2 + \frac{K^1 N_1 N_2}{2\pi N_{12}} a^3 \wedge a^1 \)

\[ p = \frac{k}{4\pi^2} \frac{N_1 N_2}{N_{12}}, \quad k \in \mathbb{Z} \]

\[ k \rightarrow k + K^1 + K^3 \]
Level Identification due to Redundancy under Level Shifting Operation

\[
\frac{K^3 N_1 N_2}{4\pi^2 N_{12}} \int_{\mathcal{M}^3} d(a^1 \wedge a^2) = N_3 \times \mathbb{Z},
\]

\[
\frac{K^1 N_1 N_2}{4\pi^2 N_{12}} \int_{\mathcal{M}^3} d(a^2 \wedge a^3) = N_1 \times \mathbb{Z},
\]

\[
\frac{K^1 N_1 N_2}{4\pi^2 N_{12}} \int_{\mathcal{M}^3} d(a^3 \wedge a^1) = N_2 \times \mathbb{Z}.
\]

Thus the minimal shift is given by:

\[
k \rightarrow k + K^1 + K^3
\]

\[
k \rightarrow k + N_{123}
\]
TQFT Hierarchy of Reducible SPT Phases

\[ \mathcal{H}^4(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots, U(1)) = \prod_{I<J} (\mathbb{Z}_{N_{II}})^2 \times \prod_{I<J<K} (\mathbb{Z}_{N_{IJK}})^2 \times \prod_{I<J<K<L} \mathbb{Z}_{N_{IJKL}} \]

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TQFT as a Field Theory Definition of Chen-Lu-Vishwanath Decoration Picture

- \( G = G_1 \times G_2 \), put (decorate) 2D \( G_2 \) SPT on 2D \( G_1 \) domain wall, and proliferate latter, leads to a 3D SPT with \( G_1 \times G_2 \). (Chen, Lu, Vishwanath, Nature Comm. 2014)

- Using our TQFT to understand it:

\[
S = \frac{i}{2\pi} \sum_1^2 \int b^I \wedge d\alpha^I + ip \int a^1 \wedge a^2 \wedge d\alpha^2 + \frac{i}{2\pi} \int b^2 \wedge dA \quad p = \frac{k}{4\pi^2} \frac{N_1 N_2}{N_{12}}
\]

Domain wall: \( a^1 = \frac{2\pi k_1}{N_1} \)

\[
\boxed{S[A] = \frac{i}{4\pi} \int_D (Z_{N_1}) \frac{2N_2 k_1 k}{N_{12}} A \wedge dA + \int dx^4 (\partial_\mu \theta - N_2 A_\mu)^2}
\]

even integer
Answers to the Questions:

- **Q:** How to derive TQFT?
  **A:** Physically derive from Phenomenological Ginzburg-Landau. (not hand-waving guess!)

- **Q:** How to impose global symmetry?
  **A:** Either modifying compactification of 2-form gauge fields or twisting symmetry. (Sym. twist is not shown in this talk.)

- **Q:** How to determine the level quantization?
  **A:** Several consistent conditions due to symmetry. If symmetry is not required, p=0.
Conclusions

- Topological Phases of Matter incl. SPT
- TQFT of SPT in one dimension (Haldane phase)
- TQFT of SPT in two dimensions (IQHE of bosons)
- **Brief** introduction to partial results of TQFT of SPT with Abelian symmetry in three dimensions.