The breakdown of the topological classification $\mathbb{Z}$ for gapped phases of noninteracting fermions by quartic interactions

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Main results

Introduction

Strategy

Examples

Application to the surfaces of SnTe

Summary

Appendices
Main results

Breakdown of the tenfold way (noninteracting topological classification of fermions) by quartic contact interactions:

<table>
<thead>
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<th>Class</th>
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Breakdown of the three-dimensional AII+$R$ topological crystalline classification $\mathbb{Z}$ to $\mathbb{Z}_8$ by quartic contact interactions.
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The tenfold way applies to noninteracting fermions that realize a noninteracting manybody ground state, a Slater determinant, such that,

(i) it is unique and it is separated from all excitations by a gap, when space is boundaryless,

(ii) while it supports gapless excitations that are extended along but bounded away from any boundary of space otherwise.
Comment 1: This definition does not require translation invariance in the direction parallel to a boundary. In particular, it applies when static impurities are present.

Comment 2: This definition implies a notion of topology since two spectral properties (the existence of gapless and extended modes) depend on the topological distinction between manifolds without (e.g., sphere, torus) and with (e.g., cube, cylinder) boundaries.
Introduction: Chronological examples

2. Chiral edge states along the boundary of an heterojunction displaying the IQHE (Laughlin 1981; Halperin 1982).
3. Chiral Majorana zero modes along the boundary of a two-dimensional \((p + ip)\) superconductor (Read and Green 2000).
4. A pair of helical zero modes along the boundary of a two-dimensional \(\mathbb{Z}_2\) topological insulator (Kane and Mele 2005).
5. The tenfold way (Schnyder, Ryu, Furusaki, and Ludwig; Kitaev 2008):

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Robustness of the gapless extended boundary states arises iff certain symmetries hold. The distinct combinations of symmetries produce 10 symmetry classes. These symmetries are local in the table

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but this need not be so. Crystalline topological insulators are noninteracting fermions that support gapless extended boundary states iff crystalline symmetries hold. An all inclusive terminology is

Short-ranged entangled (SRE) topological phases of fermionic matter \iff Symmetry-protected topological (SPT) phases of fermionic matter

C. Mudry (PSI)
Examples of diagnostic for SRE topological phases

For band insulators on a torus $\mathcal{H} = \bigoplus_{\mathbf{k} \in \text{BZ}} \mathcal{H}(\mathbf{k})$. Single-particle observables for topological SRE phases can then be chosen to be

- Polyacetylene:

  $$\mathcal{H}(k) := \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix}, \quad \nu = \frac{i}{2\pi} \int_{-\pi}^{+\pi} dk \text{tr} \left( q^\dagger \partial_k q \right)(k).$$

- IQHE for a Chern band insulator:

  $$\mathcal{H}(k) := \sum_a |u^a(k)\rangle \langle u^a(k)|, \quad w_{\mu}^{ab}(k) := \langle u^a(k)| \partial_{k^\mu} | u^b(k) \rangle, \quad \nu = \frac{i}{2\pi} \int_{-\pi}^{+\pi} d^2k \text{ tr} \left[ F_{12}(k) \right]_{\text{occupied bds}}.$$

- $\mathbb{Z}_2$ topological insulator in two-dimensional space:

  $$\mathcal{H}(k) := \sum_a |u^a(k)\rangle \langle u^a(k)|, \quad w_{\mu}^{ab}(k) := \langle u^a(-k)|\Theta| u^b(+k) \rangle, \quad \nu = \prod_{K=\mathbb{Z} \in \text{BZ}} \text{ Pf } [w(K)]_{\text{occupied bds}}.$$
The problem is that such observables are not useful in the presence of static disorder or in the presence of interactions. Alternative observables can be constructed in the presence of static disorder. The only observable that I understand in the presence of static disorder and interactions is the Hall conductivity averaged over twisted boundary conditions, Niu and Thouless 1984; Niu, Thouless, and Wu 1985

\[
C = -\frac{i}{2\pi} \int_0^{2\pi} d\phi \int_0^{2\pi} d\varphi \left[ \left\langle \frac{\partial \psi}{\partial \phi} \bigg| \frac{\partial \psi}{\partial \varphi} \right\rangle - \left\langle \frac{\partial \psi}{\partial \varphi} \bigg| \frac{\partial \psi}{\partial \phi} \right\rangle \right]
\]

with \( \psi \) the manybody ground state obeying twisted boundary conditions.

If \( C \) is an integer in the noninteracting limit, and if the interactions are not too strong (compared to the band gap), then \( C \) is unchanged by interactions. This is so because there is no backscattering allowed among the chiral edge states.
Laughlin 1981: The Hall conductivity must be rational and if it is not an integer, the ground state manifold must be degenerate and support fractionally charged excitations.

Halperin 1982: Chiral edges are immune to backscattering within each traffic lane.

Integer Quantum Hall Effect

Fractional Quantum Hall Effect
This argument (no backscattering) is tied to the boundary being a one-dimensional manifold.

Question: Are all the topological entries of the tenfold way robust to manybody interactions?
Partial answer for strong topological insulators

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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>Cl</td>
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<td>−1</td>
<td>1</td>
<td>$R_{7-d}$</td>
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<td>0</td>
<td>$\mathbb{Z}_4$</td>
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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_{32}$</td>
</tr>
</tbody>
</table>

Fidkowski and Kitaev 2010; ...
Kitaev 2011 (unpublished); Fidkowski, Chen, and Vishwanath 2014; ...
Tang and Wen 2012
Wang and Senthil 2014
Partial answer for crystalline topological insulators

There also are examples of crystalline topological insulators that are unstable to quartic interactions:

- \( \mathbb{Z} \rightarrow \mathbb{Z}_8 \) for DIII+R in 2d (Yao and Ryu 2013; Qi 2013)
- topological insulators with inversion symmetry (You and Xu 2014).
- \( \mathbb{Z} \rightarrow \mathbb{Z}_8 \) for All+R in 3d (Isobe and Fu 2015; Yoshida and Furusaki; this work 2015)
Many different approaches were used to reach these conclusions. Is there an approach, perhaps a simpler one, that reproduces all known results and extends readily to all entries of the tenfold way?
1. Main results

2. Introduction

3. Strategy

4. Examples

5. Application to the surfaces of SnTe

6. Summary

7. Appendices
Strategy

The strategy that we use to study the robustness of $\nu$ boundary modes to quartic contact fermion-fermion interactions is related to the approach advocated by You and Xu 2014 and (unpublished) Kitaev 2015.

It consists of three steps.
Step 1

A noninteracting topological phase is represented by the manybody ground state of a massive Dirac Hamiltonian with a matrix dimension that depends on $\nu$. 
Step 2

A Hubbard-Stratonovich transformation is used to trade a generic quartic contact interaction in favor of dynamical Dirac mass-like bilinears coupled to their conjugate fields (that will be called dynamical Dirac masses). These dynamical Dirac masses may violate any symmetry constraint other than the particle-hole symmetry (PHS).
Step 3

The $\nu$ boundary modes that are coupled with a suitably chosen subset of dynamical Dirac masses are integrated over. The resulting dynamical theory on the $(d - 1)$-dimensional boundary is a bosonic one, a quantum nonlinear sigma model (QNLSM) in $[(d - 1) + 1]$ space and time, with a target space that depends on $\nu$.

These QNLSM can have the following phase diagrams:

Without topological term

$d \leq 2$

\[ g = 0 \]

With topological term

$d > 2$

\[ g = 0 \]

\[ g_c \]
Step 4

The reduction pattern is then obtained by identifying the smallest value of $\nu$ for which:

- This QNLSM cannot be augmented by a topological term that is compatible with local equations of motion.
- The quantum fluctuations driven by the interactions have restored all the symmetries broken by the saddle point.
Comment 1: The question that we address in this paper is whether the topological classification of noninteracting fermions is reduced by interactions. A complete classification of fermionic SRE phases (combined with that for the bosonic SRE phases) is beyond the scope of our approach.

Comment 2: The presence or absence of topological terms in the relevant QNLSM is determined by the topology of the spaces of boundary dynamical Dirac masses, i.e., the topology of classifying spaces. Now, K-theory provides a systematic way to study the topology of the classifying space.

Comment 3: This is why the same approach that was used to obtain the tenfold way of noninteracting fermions can be relied on to deduce a classification of topological short-range entangled (SRE) phases for interacting fermions.
Main results

Introduction

Strategy

Examples

Application to the surfaces of SnTe

Summary

Appendices
Example: Symmetry class AII in $d = 3$

Consider the single-particle static massive Dirac Hamiltonian [this is THE Dirac Hamiltonian (Dirac 1928)]

$$\mathcal{H}^{(0)}(x) := -i\partial_x X_{21} - i\partial_y X_{11} - i\partial_z X_{02} + m(x) X_{03}, \quad (1a)$$

where

$$X_{\mu\mu'} := \sigma_\mu \otimes \tau_{\mu'} \quad (1b)$$

for $\mu, \mu' = 0, 1, 2, 3$. Because

$$\mathcal{H}^{(0)}(x) = +T \mathcal{H}^{(0)}(x) T^{-1}, \quad T := iX_{20} K, \quad (1c)$$

we interpret this Hamiltonian as realizing a noninteracting topological insulator in the three-dimensional symmetry class AII.
The static domain wall in the mass (Jackiw and Rebbi 1976)

\[ m(x, y, z) = m_{\infty} \text{sgn}(z) \]  

binds a zero mode to the boundary \( z = 0 \) that is annihilated by the boundary single-particle Hamiltonian

\[
\mathcal{H}_{\text{bd}}^{(0)}(x, y) = -i\partial_x \sigma_2 - i\partial_y \sigma_1 \\
= \mathcal{T}_{\text{bd}} \mathcal{H}_{\text{bd}}^{(0)}(x, y) \mathcal{T}_{\text{bd}}^{-1},
\]

where

\[
\mathcal{T}_{\text{bd}} := i\sigma_2 K.
\]
The boundary dynamical Dirac Hamiltonian

\[ \mathcal{H}^{(\text{dyn})}_{\text{bd}}(\tau, x, y) = -i \partial_x \sigma_2 - i \partial_y \sigma_1 + M(\tau, x, y) \sigma_3 \]  

(3)

belongs to the symmetry class A, as the Dirac mass \( M \sigma_3 \) breaks TRS unless \( M(-\tau, x, y) = -M(\tau, x, y) \).

The space of normalized boundary dynamical Dirac mass matrices \( \{ \pm \sigma_3 \} \) is homeomorphic to the space of normalized Dirac mass matrices in the symmetry class A

\[ V_{\nu=1} = \bigcup_{k=0}^{1} U(\nu)/[U(k) \times U(\nu - k)] \]  

(4)

when space is two dimensional.

The domain wall in imaginary time (the instanton)

\[ M(\tau, x, y) = M_\infty \text{sgn}(\tau) \]  

(5)

prevents the gapping of the boundary zero mode.
We conclude that the noninteracting topological classification $\mathbb{Z}_2$ of three-dimensional insulators in the symmetry class AII is robust to the effects of interactions under the assumption that only fermion-number-conserving interacting channels are included in the stability analysis. The logic used to reach this conclusion is summarized by Table

<table>
<thead>
<tr>
<th>$D$</th>
<th>$\pi_D(C_0)$</th>
<th>$\nu$</th>
<th>Topological obstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>1</td>
<td>Domain wall</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

once the line corresponding to $\nu = 1$ has been identified. This line is fixed by the first integer $D$ that accommodates a non-trivial entry for the corresponding homotopy group.

Moreover, one verifies by introducing a BdG (Nambu) grading that this robustness extends to interaction-driven dynamical superconducting fluctuations.
Example: Symmetry class AII in $d = 3$ with reflection symmetry $R$

We consider again the bulk, boundary, and dynamical boundary Hamiltonian defined in Eqs. (1)–(4). We observe that the single-particle Hamiltonian (1a) has the symmetry

$$\mathcal{R}_x \mathcal{H}^{(0)}(-x, y, z) (\mathcal{R}_x)^{-1} = +\mathcal{H}^{(0)}(x, y, z),$$

(6a)

where

$$\mathcal{R}_x := iX_{10}, \quad \mathcal{R}_x^2 = -1, \quad [\mathcal{T}, \mathcal{R}_x] = 0,$$

(6b)

in addition to the TRS (1c).
The presence of the additional reflection symmetry allows one to define a mirror Chern number ($n_+ \in \mathbb{Z}$) for the sector with the eigenvalue $\mathcal{R}_x = +i$ on the two-dimensional mirror plane ($k_x = 0$) in the three-dimensional Brillouin zone (Hsieh, Lin, Liu, Duan, Bansil, and Fu 2012).

Thus, the $\nu$ linearly independent zero modes that follow from tensoring the single-particle Hamiltonian (1a) with the $\nu \times \nu$ unit matrix along the domain wall (2a) are stable to strong one-body perturbations on the boundary that preserve the reflection symmetry. (If we forget the reflection symmetry and keep only the TRS, it is only the parity of $\nu$ that is stable to strong one-body perturbations on the boundary.)
If we only consider dynamical masses that preserve the fermion-number $U(1)$ symmetry, the space of normalized boundary dynamical Dirac mass matrices after tensoring the boundary dynamical Dirac Hamiltonian (4) with the unit $\nu \times \nu$ matrix is homeomorphic to the space of normalized Dirac masses in the symmetry class $A$

$$V_{\nu} = \bigcup_{k=0}^{\nu} U(\nu)/[U(k) \times U(\nu - k)]$$

when space is two dimensional.

The limit $\nu \to \infty$ of these spaces is the classifying space $C_0$. 
Integrating the boundary Dirac fermions delivers a QNLSM in (2+1)-dimensional space and time. In order to gap out dynamically the boundary zero modes without breaking the symmetries, this QNLSM must be free of topological obstructions. We construct explicitly the spaces for the relevant normalized boundary dynamical Dirac mass matrices \( [M(\tau, x, y) \text{ in Eq. (3)}) \) of dimension \( \nu = 2^n \) with \( n = 0, 1, 2, 3 \) in the following.
Case $\nu = 1$:

There is a topological obstruction of the \textbf{domain-wall} type as the target space is

$$S^0 = \{\pm 1\} \quad (8)$$

and $\pi_0(S^0) \neq 0$. 
Case \( \nu = 2 \):

There is a topological obstruction of the monopole type as the target space is

\[
S^2 = \{ c_1 X_1 + c_2 X_2 + c_3 X_3 \mid c_1^2 + c_2^2 + c_3^2 = 1 \}
\]  (9)

and \( \pi_2(S^2) = \mathbb{Z} \).
Case \( \nu = 4 \):

There is a topological obstruction of the WZ type as the target space is

\[
S^4 = \left\{ c_1 X_{13} + c_2 X_{23} + c_3 X_{33} + c_4 X_{01} + c_5 X_{02} \left| \sum_{i=1}^{5} c_i^2 = 1, \; c_i \in \mathbb{R} \right. \right\}
\]

(10)

and \( \pi_4(S^4) = \mathbb{Z} \).
Case $\nu = 8$:

There is no topological obstruction as one can find more than five pairwise anticommuting matrices such as the set

$$\{X_{133}, X_{233}, X_{333}, X_{013}, X_{023}, X_{001}, X_{002}\}.$$  
(11)
Reduction from $\mathbb{Z}$ to $\mathbb{Z}_8$ due to interactions for the topologically equivalent classes of the three-dimensional topological insulators with time-reversal and reflection symmetries ($\text{AII} + R$). We denote by $V_\nu$ the space of $\nu \times \nu$ normalized Dirac mass matrices in boundary ($d = 2$) Dirac Hamiltonians belonging to the symmetry class A. The limit $\nu \to \infty$ of these spaces is the classifying space $C_0$. The second column shows the stable $D$-th homotopy groups of the classifying space $C_0$. The third column gives the number $\nu$ of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

<table>
<thead>
<tr>
<th>$D$</th>
<th>$\pi_D(C_0)$</th>
<th>$\nu$</th>
<th>Topological obstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>1</td>
<td>Domain wall</td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{Z}$</td>
<td>2</td>
<td>Monopole</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}$</td>
<td>4</td>
<td>WZ term</td>
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</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}$</td>
<td>8</td>
<td>None</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
This $\mathbb{Z}_8$ classification is unchanged if all boundary dynamical masses that break the fermion-number $U(1)$ symmetry are accounted for. The corresponding target spaces for the boundary dynamical masses and their topological obstructions are derived by extending the single-particle Hamiltonian [Eq. (3)] to a BdG Hamiltonian

$$
\mathcal{H}^{(\text{dyn})}_{\text{bd}} = (-i\partial_x \sigma_2 \otimes \rho_3 - i\partial_y \sigma_1 \otimes \rho_0) \otimes l_{\nu \times \nu} + \gamma'(\tau, x, y),
$$

(12a)

where $\rho_0$ and $\rho_\mu$ are unit $2 \times 2$ and Pauli matrices, respectively, acting on the particle-hole (Nambu) space, $l_{\nu \times \nu}$ is the unit $\nu \times \nu$ matrix, and the particle-hole symmetry is given by $C = \rho_1 K$. In this case, the target spaces of the QNLSM made of normalized boundary dynamical Dirac mass matrices $\gamma'$ of dimension $\nu = 2^n$ with $n = 0, 1, 2, 3$ are modified as listed in the following with the notation

$$
X_{\mu \mu' \mu'' \mu'''...} = \sigma_\mu \otimes \rho_\mu' \otimes \tau_\mu'' \otimes \tau_\mu''' \ldots.
$$

(12b)
The relevant homotopy groups are given in

<table>
<thead>
<tr>
<th>$D$</th>
<th>$\pi_D(R_0)$</th>
<th>$\nu$</th>
<th>Topological obstruction</th>
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</thead>
<tbody>
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</tr>
<tr>
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<td>$\mathbb{Z}$</td>
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<td>WZ term</td>
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<tr>
<td>8</td>
<td>$\mathbb{Z}$</td>
<td>8</td>
<td>None</td>
</tr>
</tbody>
</table>

We note that these target spaces are closed under the global $U(1)$ transformation generated by $\rho_3$. 


5. Application to the surfaces of SnTe
Application to the surfaces of SnTe

The crystal SnTe is a three-dimensional topological crystalline insulator protected by time-reversal and reflection symmetries (AII + R).

SnTe supports four Dirac cones on the [001] surface and six Dirac cones on the [111] surface.

If strong interaction effects are present, we expect the following.

- On the [001] surface, the $\nu = 4$ phase described by a QNLSM with a WZ term should be realized.
- On the [111] surface, the $\nu = 6 = 4 + 2$ phase should be realized, whereby the effective field theory is that of a QNLSM with a WZ term for 4 out of the six surface Dirac cones and that of a QNLSM with a topological term arising from a gas of monopoles for the remaining two surface Dirac cones.
The breakdown of the topological classification \( Z \) for gapped phases of noninteracting fermions by quartic interactions.
Breakdown of the tenfold way (noninteracting topological classification of fermions) by quartic contact interactions:

<table>
<thead>
<tr>
<th>Class</th>
<th>(T)</th>
<th>(C)</th>
<th>(\Gamma_5)</th>
<th>(V_d)</th>
<th>(d=1)</th>
<th>(d=2)</th>
<th>(d=3)</th>
<th>(d=4)</th>
<th>(d=5)</th>
<th>(d=6)</th>
<th>(d=7)</th>
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<tbody>
<tr>
<td>A</td>
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<td>0</td>
<td>(C_{0+d})</td>
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<td>0</td>
<td>(\mathbb{Z})</td>
<td>0</td>
<td>(\mathbb{Z})</td>
<td>0</td>
<td>(\mathbb{Z})</td>
</tr>
<tr>
<td>AIII</td>
<td>0</td>
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<td>1</td>
<td>(C_{1+d})</td>
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<td>(\mathbb{Z}_8)</td>
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<td>(\mathbb{Z}_{16})</td>
<td>0</td>
<td>(\mathbb{Z}_{32})</td>
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</tr>
<tr>
<td>AI</td>
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<td>0</td>
<td>(R_{0-d})</td>
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<td>0</td>
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<td>(\mathbb{Z})</td>
</tr>
<tr>
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<td>+1</td>
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<td>(R_{1-d})</td>
<td>(\mathbb{Z}_8, \mathbb{Z}_4)</td>
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<td>0</td>
<td>0</td>
<td>(\mathbb{Z}_{16}, \mathbb{Z}_8)</td>
<td>0</td>
<td>(\mathbb{Z}_2)</td>
<td>(\mathbb{Z}_2)</td>
</tr>
<tr>
<td>D</td>
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<td>0</td>
<td>(R_{2-d})</td>
<td>(\mathbb{Z}_2)</td>
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<td>0</td>
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<td>0</td>
<td>(\mathbb{Z})</td>
<td>0</td>
<td>(\mathbb{Z}_2)</td>
</tr>
<tr>
<td>DIII</td>
<td>−1</td>
<td>+1</td>
<td>1</td>
<td>(R_{3-d})</td>
<td>(\mathbb{Z}_2)</td>
<td>(\mathbb{Z}_2)</td>
<td>(\mathbb{Z}_{16})</td>
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<td>(\mathbb{Z}_2)</td>
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<td>(\mathbb{Z})</td>
</tr>
<tr>
<td>CII</td>
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<td>−1</td>
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<td>(R_{5-d})</td>
<td>(\mathbb{Z}_2, \mathbb{Z}_2)</td>
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<td>(\mathbb{Z}_2)</td>
<td>(\mathbb{Z}_2)</td>
<td>(\mathbb{Z}<em>{16}, \mathbb{Z}</em>{16})</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
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</tr>
<tr>
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</tr>
</tbody>
</table>

Breakdown of the three-dimensional All+\(R\) topological crystalline classification \(\mathbb{Z}\) to \(\mathbb{Z}_8\) by quartic contact interactions.
1. Main results
2. Introduction
3. Strategy
4. Examples
5. Application to the surfaces of SnTe
6. Summary
7. Appendices
The classifying spaces and their homotopy groups are

<table>
<thead>
<tr>
<th>Class</th>
<th>$T$</th>
<th>$C$</th>
<th>$\Gamma$</th>
<th>Extension</th>
<th>$V_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
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<td>0</td>
<td>0</td>
<td>$Cl_d \rightarrow Cl_{d+1}$</td>
<td>$C_{0+d}$</td>
</tr>
<tr>
<td>All</td>
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<td>$Cl_{d+1} \rightarrow Cl_{d+2}$</td>
<td>$C_{1+d}$</td>
</tr>
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<td>$Cl_{0,d+2} \rightarrow Cl_{1,d+2}$</td>
<td>$R_{0-d}$</td>
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<td>$Cl_{d+1,2} \rightarrow Cl_{d+1,3}$</td>
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<td>$R_{2-d}$</td>
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<td>$R_{3-d}$</td>
</tr>
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<td>All</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>$Cl_{2,d} \rightarrow Cl_{3,d}$</td>
<td>$R_{4-d}$</td>
</tr>
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<td>$Cl_{d+3,0} \rightarrow Cl_{d+3,1}$</td>
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<td>$Cl_{d+2,0} \rightarrow Cl_{d+2,1}$</td>
<td>$R_{6-d}$</td>
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<tr>
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<td>$Cl_{d+2,1} \rightarrow Cl_{d+2,2}$</td>
<td>$R_{7-d}$</td>
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<table>
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<tr>
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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
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</table>
Main results (AZ stands for Altland and Zirnbauer)

Case $d = 1$

Five AZ symmetry classes

Three AZ symmetry classes

Two AZ symmetry classes

(A) $N=1$

(B) $N=2$ and so on

(C) $N=2$ and so on

Case $d = 3$

Five AZ symmetry classes

Three AZ symmetry classes

Two AZ symmetry classes

(A) $N=1$

(B) $N=2$ and so on

(C) $N=2$ and so on
Recent related works


Main results

Introduction

Strategy

Examples

Application to the surfaces of SnTe

Summary

Appendices

C. Mudry (PSI)
The Dirac equation: an application of Clifford algebra


\[
\left( \beta mc^2 + c(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3) \right) \psi(x, t) = i\hbar \frac{\partial \psi(x, t)}{\partial t}
\]

where

\[
\{ \alpha_i, \alpha_j \} = 2 \delta_{ij}, \quad \{ \alpha_i, \beta \} = 0, \quad i, j = 1, 2, 3, \quad \beta^2 = 1.
\]
1931: Dirac introduces topology in physics

Quantised Singularities in the Electromagnetic Field

P. A. M. Dirac

1932-1939: Tamm and Schockley surface states


P-Y. Chang, C. Mudry, and S. Ryu 2014

![Graph showing energy bands for a superconductor in a cylindrical geometry.]

Direct sum of a $p_x + ip_y$ and of a $p_x - ip_y$ BdG superconductor in a cylindrical geometry.
Anderson localization before 1981


In a noninteracting many-body setting, there are three possible phases:

- The metallic phase: \( \sigma = \infty \) at \( T = 0 \).
- The insulating phase: \( \sigma = 0 \) at \( T = 0 \).
- A quantum critical phase: \( \sigma = \sigma_c \) at \( T = 0 \).
1953: Dyson’s exception

1963: The threefold way for random matrices


\[ P(\theta_1, \cdots, \theta_N) \propto \prod_{1 \leq j < k \leq N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta, \quad \beta = 1, 2, 4. \]
1976: Jackiw and Rebbi introduce Fermion number fractionalization


1981: Nielsen-Ninomiya theorem

1981: Jackiw and Rossi introduce a localized Majorana zero mode in a two-dimensional relativistic superconductor


L. Fu and C. L. Kane, “Superconducting Proximity Effect and Majorana Fermions at the Surface of a Topological Insulator,” Phys. Rev. Lett. 100, 096407 (2008) have proposed to realize this localized Majorana zero mode on the core of a superconducting vortex induced by proximity effect on the surface of a strong $\mathbb{Z}_2$ topological insulator.
Anderson localization after 1981

The discovery of the Integer Quantum Hall effect, Klitzing, Dorda, and Pepper (1980), lead to the understanding that there are topologically distinct insulating phases separated by critical point at which a plateau transition takes place.

Example: Graphene deposited on SiO$_2$/Si, $T=1.6$ K and $B=9$ T (inset $T=30$ mK): $\nu = \pm 2, \pm 6, \pm 10, \cdots = \pm 2(2n+1), \ n \in \mathbb{N}$

after Zhang et al., 2005.
At integer fillings of the Landau levels, the noninteracting ground state is unique and the screened Coulomb interaction $V_{\text{int}}$ can be treated perturbatively, as long as transitions between Landau levels or outside the confining potential $V_{\text{conf}}$ along the magnetic field are suppressed by the single-particle gaps:

$$V_{\text{int}} \ll \hbar \omega_c \ll V_{\text{conf}}, \quad \omega_c = e B / (m c).$$
1981-1982: Laughlin and Halperin introduce the bulk-edge correspondence in the Quantum Hall Effect

**Laughlin 1981:** The Hall conductivity must be rational and if it is not an integer, the ground state manifold must be degenerate and support fractionally charged excitations.

**Halperin 1982:** Chiral edges are immune to backscattering within each traffic lane.

---

Integer Quantum Hall Effect

Fractional Quantum Hall Effect
1982: TKNN relate linear response to topology in the Quantum Hall Effect

Thouless, Kohmoto, Nightingale, and den Nijs, (1982),
Avron, Seiler, and Simon (1983); Simon (1983),
Niu and Thouless (1984); Niu, Thouless, and Wu (1985)

The Hall conductance is proportional to the first Chern number

$$C = -\frac{i}{2\pi} \int_0^{2\pi} d\phi \int_0^{2\pi} d\varphi \left[ \left\langle \frac{\partial \psi}{\partial \phi} \right| \frac{\partial \psi}{\partial \varphi} \right\rangle - \left\langle \frac{\partial \psi}{\partial \varphi} \right| \frac{\partial \psi}{\partial \phi} \right\rangle$$

with $\psi$ the many-body ground state obeying twisted boundary conditions.
1983-1985: Khmelnitskii and Pruisken introduce the scaling theory of the Integer Quantum Hall Effect

A topological term modifies the scaling analysis of the gang of four:
Khmelnitskii 1983
Pruisken 1985

\[
\frac{d \ln g}{d \ln L} = \begin{cases} 
\text{d} = 2, & \phi = 0 \\
\text{d} = 2, & \phi = \pi 
\end{cases}
\]

C. Mudry (PSI)
1983-1984: Haldane introduces the $\theta$ term for spin chains and Witten achieves non-Abelian bosonization


$O(3) \text{ NLSM}$

1988: Haldane model for a Chern band insulator

1994: Random Dirac fermions in two-dimensional space

1997: The tenfold way for random matrices

<table>
<thead>
<tr>
<th>Cartan label</th>
<th>T</th>
<th>C</th>
<th>S</th>
<th>Hamiltonian</th>
<th>G/H (ferm. NLSM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (unitary)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>U(N)</td>
<td>U(2n)/U(n) × U(n)</td>
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<td>AI (orthogonal)</td>
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<td>Sp(2n)/Sp(n) × Sp(n)</td>
</tr>
<tr>
<td>AII (symplectic)</td>
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<td>O(2n)/O(n) × O(n)</td>
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<tr>
<td>AIII (ch. unit.)</td>
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<td>0</td>
<td>1</td>
<td>U(N + M)/U(N) × U(M)</td>
<td>U(n)</td>
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<tr>
<td>BDI (ch. orth.)</td>
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<td>+1</td>
<td>1</td>
<td>O(N + M)/O(N) × O(M)</td>
<td>U(2n)/Sp(2n)</td>
</tr>
<tr>
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<td>Sp(N + M)/Sp(N) × Sp(M)</td>
<td>U(2n)/O(2n)</td>
</tr>
<tr>
<td>D (BdG)</td>
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<td>SO(2N)</td>
<td>O(2n)/U(n)</td>
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<td>C (BdG)</td>
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<td>0</td>
<td>Sp(2N)</td>
<td>Sp(2n)/U(n)</td>
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<tr>
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<td>1</td>
<td>SO(2N)/U(N)</td>
<td>O(2n)</td>
</tr>
<tr>
<td>CI (BdG)</td>
<td>+1</td>
<td>−1</td>
<td>1</td>
<td>Sp(2N)/U(N)</td>
<td>Sp(2n)</td>
</tr>
</tbody>
</table>

The column entitled “Hamiltonian” lists, for each of the ten symmetry classes, the symmetric space of which the quantum mechanical time-evolution operator \( \exp(it\mathcal{H}) \) is an element. The last column entitled “G/H (ferm. NLSM)” lists the (compact sectors of the) target space of the NL\(\sigma\)M describing Anderson localization physics at long wavelength in this given symmetry.
1998-2000: Brouwer et al. establish that there are five symmetry class that display quantum criticality in disordered quasi-one-dimensional wires

The “radial coordinate” of the transfer matrix $\mathcal{M}$ from the Table below makes a Brownian motion on an associated symmetric space.

<table>
<thead>
<tr>
<th>Class</th>
<th>TRS</th>
<th>SRS</th>
<th>$m_o$</th>
<th>$m_l$</th>
<th>$D$</th>
<th>$\mathcal{M}$</th>
<th>$\mathcal{H}$</th>
<th>$\delta g$</th>
<th>$\langle - \ln g \rangle$</th>
<th>$\rho(\varepsilon)$ for $0 &lt; \varepsilon \tau_c \ll 1$</th>
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<tbody>
<tr>
<td>O</td>
<td>Yes</td>
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<td>1</td>
<td>2</td>
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<td>AI</td>
<td>$-2/3$</td>
<td>$2L/(\gamma \ell)$</td>
<td>$\rho_0$</td>
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<tr>
<td>U</td>
<td>No</td>
<td>Y(N)</td>
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<tr>
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<td>N</td>
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<td>Y</td>
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<td>0</td>
<td>2</td>
<td>AlI</td>
<td>BDI</td>
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<td>N</td>
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<td>0</td>
<td>2</td>
<td>AllI</td>
<td>Cl</td>
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<td>$2m_0 L/(\gamma \ell)$</td>
<td>$(\pi \rho_0 / 3) (\varepsilon \tau_c)^3 \ln</td>
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<td>Y</td>
<td>Y</td>
<td>2</td>
<td>2</td>
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<td>$(\pi \rho_0 / 2)</td>
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<td>4</td>
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<tr>
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<td>DIII</td>
<td>$+2/3$</td>
<td>$4 \sqrt{L/(2\pi \gamma \ell)}$</td>
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<td>$+1/3$</td>
<td>$4 \sqrt{L/(2\pi \gamma \ell)}$</td>
<td>$\pi \rho_0 /</td>
</tr>
</tbody>
</table>
2000: Read and Green introduce the chiral $p$-wave topological superconductor

2005: Kane and Mele introduce the strong $\mathbb{Z}_2$ topological insulator

The spin-orbit coupling is ignored in the QHE as the breaking of time-reversal symmetry provides the dominant energy scale. C. L. Kane and E. J. Mele, “$\mathbb{Z}_2$ Topological Order and the Quantum Spin Hall Effect,” Phys. Rev. Lett. 95, 146802 (2005): Combine a pair of time-reversed Haldane models with a small Rashba coupling and find protected helical edge states.

\( \Delta \)
2008: The tenfold way for topological insulators and superconductors

complex case:

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</table>

real case:

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<td>Z(_2)</td>
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<td>Z(_2)</td>
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<td>0</td>
<td>2Z</td>
<td>...</td>
</tr>
</tbody>
</table>
1 Main results

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3 Strategy

4 Examples

5 Application to the surfaces of SnTe

6 Summary

7 Appendices
Definition and question to be addressed

Consider the random Dirac Hamiltonian in $d$-dimensional space of rank $r$ given by

$$\mathcal{H}_{d,r} = \sum_{i=1}^{d} \alpha_i \frac{\partial}{i \partial x_i} + \mathcal{M}(\mathbf{x}) + \cdots, \quad \{\alpha_i, \alpha_j\} = 2 \delta_{ij}, \quad i, j = 1, \cdots, d,$$

where

$$\frac{1}{r} \text{tr}[\mathcal{M}(\mathbf{x}) - \bar{\mathcal{M}}][\mathcal{M}(\mathbf{y}) - \bar{\mathcal{M}}] =: g^2 e^{-|\mathbf{x} - \mathbf{y}|/\xi_{\text{dis}}},$$

with all higher cumulants vanishing.

Is the single-particle energy eigenstate at $\varepsilon = 0$ delocalized or localized?

A necessary but not sufficient condition for localization is that a Dirac mass matrix is permitted i.e., $\mathcal{M}(\mathbf{x}) \neq 0$. 
Why “necessary but not sufficient”?

Choose \( d = 1 \) and \( r = 2 \):

\[
\mathcal{H}_{d=1,r=2} = \tau_1 [-i \partial + A_1(x)] + \tau_0 A_0(x) + \tau_2 m_2(x) + \tau_3 m_3(x).
\]

There is a single localized phase.

Choose \( d = 2 \) and \( r = 2 \):

\[
\mathcal{H}_{d=2,r=2} = \tau_1 [-i \partial_1 + A_1(x)] + \tau_2 [-i \partial_2 + A_2(x)] + \tau_0 A_0(x) + \tau_3 m(x).
\]

There are two topological distinct phases and a critical line separating them if \( m(x) = 0 \).

Choose \( d = 3 \) and \( r = 2 \):

\[
\mathcal{H}_{d=3,r=1} = \tau_1 [-i \partial_1 + A_1(x)] + \tau_2 [-i \partial_2 + A_2(x)] + \tau_3 [-i \partial_3 + A_3(x)] + \tau_0 A_0(x).
\]

Localization is prohibited.
A normalized Dirac mass matrix $\beta$ is defined by the condition

$$\{\alpha, \beta\} = 0, \quad \beta^2 = 1.$$ 

When $d = 1$ and $r = 2$, the normalized Dirac mass matrix is

$$\beta(x) = \tau_2 \cos \theta(x) + \tau_3 \sin \theta(x), \quad \tan \theta(x) := \frac{m_3(x)}{m_2(x)}.$$ 

As a space, it is the unit circle.
A normalized Dirac mass matrix \( \beta \) is defined by the condition

\[
\{ \alpha, \beta \} = 0, \quad \beta^2 = 1.
\]

When \( d = 2 \) and \( r = 2 \), the normalized Dirac mass matrix is

\[
\beta(x) = \frac{m(x)}{|m(x)|} \tau_3,
\]

As a space, it is two points.

If \( m(x) = m \), then \( \tau_{xy} = \frac{1}{2} \text{sgn} m \). The domain wall \( m(x, y) = m(x) \) with \( \lim_{x \to \pm \infty} m(x) = \mp m \) supports the boundary states

\[
\psi(x, y; k) \propto e^{iky - i\tau_3 \int_{-\infty}^{x} dx' m(x')} \begin{pmatrix} 1 \\ -i \end{pmatrix}
\]

with the dispersion

\( \varepsilon(k) = k \).

This is the minimal model for the IQHE.
A normalized Dirac mass matrix $\beta$ is defined by the condition

$$\{\alpha, \beta\} = 0, \quad \beta^2 = 1_r.$$ 

When $d = 3$ and $r = 2$, the normalized Dirac mass matrix is not allowed.
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The tenfold way for the classifying spaces when $d = 1$

Symmetry class A:

$$\mathcal{H}(k) := \tau_3 k \tau_3 A_1 + \tau_2 M_2 + \tau_1 M_1 + \tau_0 A_0, \quad V_{d=1; r=2}^A = S^1.$$ 

Symmetry class AII:

$$\mathcal{H}(k) = +\tau_2 \mathcal{H}^*(-k) \tau_2, \quad V_{d=1; r=2}^{AII} = \emptyset.$$ 

Symmetry class AI:

$$\mathcal{H}(k) = +\tau_1 \mathcal{H}^*(-k) \tau_1, \quad V_{d=1; r=2}^{AI} = S^1.$$ 

Symmetry class AIII:

$$\mathcal{H}(k) = -\tau_1 \mathcal{H}(k) \tau_1, \quad V_{d=1; r=2}^{AIII} = \{\pm \tau_2\}.$$ 

Symmetry class CII: No Dirac Hamiltonian when $r = 2$.

Symmetry class BDI:

$$\mathcal{H}(k) = -\tau_1 \mathcal{H}(k) \tau_1 = +\tau_1 \mathcal{H}^*(-k) \tau_1, \quad V_{d=1; r=2}^{BDI} = \{\pm \tau_2\}.$$
Symmetry class D:

\[ \mathcal{H}(k) = -\mathcal{H}^*(-k), \quad V^D_{d=1;r=2} = \{\pm \tau_1\}. \]

Symmetry class DIII:

\[ \mathcal{H}(k) = -\mathcal{H}^*(-k) = +\tau_2 \mathcal{H}^*(-k) \tau_2, \quad V^{DIII}_{d=1;r=2} = \emptyset. \]

Symmetry class C:

\[ \mathcal{H}(k) = -\tau_2 \mathcal{H}^*(-k) \tau_2 = \tau_3 A_1 + \tau_2 M_2 + \tau_1 M_1. \]

PHS squaring to minus unity prohibits a kinetic energy in the symmetry class C.

Symmetry class CI:

\[ \mathcal{H}(k) = -\tau_2 \mathcal{H}^*(-k) \tau_2 = +\tau_1 \mathcal{H}^*(-k) \tau_1 = \tau_2 M_2 + \tau_1 M_1. \]

PHS squaring to minus unity prohibits a kinetic energy in the symmetry class CI.
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Existence and uniqueness of normalized Dirac masses

For each symmetry class, there exists a minimum rank \( N \ni r_{\text{min}}(d) > 2 \), for which the \( d \)-dimensional Dirac Hamiltonian supports a mass matrix and below which either no mass matrix or no Dirac kinetic contribution are allowed by symmetry.

Suppose that the rank of the \( d \)-dimensional massive Dirac Hamiltonian is \( r = r_{\text{min}}(d) N \). When there exists a mass matrix squaring to unity (up to a sign) that commutes with all other symmetry-allowed mass matrices, we call it the unique mass matrix.
Existence and uniqueness of normalized Dirac masses

There are the following three cases:

(a) $\pi_0(V_d) = \{ 0 \}$: For any given mass matrix, there exists another mass matrix that anticommutes with it for $N \geq 1$.

(b) $\pi_0(V_d) = \mathbb{Z}$: There always is a unique mass matrix for $N \geq 1$.

(c) $\pi_0(V_d) = \mathbb{Z}_2$: There is a unique mass matrix when $N$ is an odd integer. However, when $N$ is an even integer, there is no such unique mass matrix.

When $N = 1$ and when the Dirac Hamiltonian has a unique mass matrix, then topologically distinct ground states are realized for different signs of its mass. When the mass is varied smoothly in space, domain boundaries where the sign-of-mass changes are accompanied by massless Dirac fermions, whose low-energy Hamiltonian is of rank $r_{\text{min}}(d)/2$. It follows that $r_{\text{min}}(d-1) = r_{\text{min}}(d)$. 
Path connectedness of the normalized Dirac masses

Case (a): $\pi_0(V) = \{0\}$

Case (b): $\pi_0(V) = \mathbb{Z}$

Case (c): $\pi_0(V) = \mathbb{Z}_2$
Network model and criticality

For each realization of a random potential perturbing the massless Dirac Hamiltonian defined in \(d\)-dimensional Euclidean space, it may be decomposed into open sets (domains) of linear size \(\xi_{\text{dis}}\). In each of these domains, the normalized Dirac masses correspond to a unique value of their zeroth homotopy group. At the boundary between domains differing by the values taken by their zeroth homotopy group, the Dirac masses must vanish. Such boundaries support zero-energy boundary states.

When it is possible to classify the elements of the zeroth homotopy group by the pair of indices \(A\) and \(B\), we may assign the letters \(A\) or \(B\) to any one of these domains as is illustrated. When the typical “volume” of a domain of type \(A\) equals that of type \(B\), quasi-zero modes undergo quantum percolation through the sample and thus establish either a critical or a metallic phase of quantum matter in \(d\)-dimensional space.
The solution for the classifying spaces is

<table>
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<tr>
<th>Class</th>
<th>$T$</th>
<th>$C$</th>
<th>$\Gamma$</th>
<th>Extension</th>
<th>$V_d$</th>
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where

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The parameters for the phase diagram: $\pi_0(V) = \{0\}$

For any of the five AZ symmetry classes with $V = C_1, R_3, R_5, R_6, R_7$, we can always choose the parameter space $(m, g) \in \mathbb{R} \times [0, \infty[$ by selecting

$$\alpha = \alpha_{\text{min}} \otimes 1_N, \quad \beta_0 := \beta_{\text{min}} \otimes 1_N,$$

and

$$\overline{\mathcal{M}(x)} =: m \beta_0,$$

$$\frac{1}{r} \text{tr} \left\{ \left[ \overline{\mathcal{M}(x)} - m \beta_0 \right] \left[ \overline{\mathcal{M}(y)} - m \beta_0 \right] \right\} =: g^2 e^{-|x-y|/\xi_{\text{dis}}}.$$
The parameters for the phase diagram: $\pi_0(V) = \mathbb{Z}$

For any of the three AZ symmetry classes with $V = C_0, R_0, R_4$, we can always choose the parameter space $(m, g) \in \mathbb{R} \times [0, \infty[$ by selecting

$$\alpha = \alpha_{\text{min}} \otimes 1_N, \quad \beta_0 := \beta_{\text{min}} \otimes 1_N,$$

and

$$\overline{M(x)} =: m \beta_0,$$

$$\frac{1}{r} \text{tr} \left\{ [\overline{M(x)} - m \beta_0][\overline{M(y)} - m \beta_0] \right\} =: g^2 \ e^{-|x-y|/\xi_{\text{dis}}}.$$
Moreover, we can also always write

\[ \mathcal{M}(\mathbf{x}) = \beta_{\text{min}} \otimes \mathcal{M}(\mathbf{x}), \]

where the condition

\[ \det[\mathcal{M}(\mathbf{x})] = 0 \]

defines the boundaries of the domains from the quantum network model, each of which can be assigned the index

\[ \nu := \frac{1}{2} \text{tr}\{\text{sgn}[\mathcal{M}(\mathbf{x})]\} \in \begin{cases} \{-1/2, +1/2\}, & \text{if } N = 1, \\ \{-1, 0, +1\}, & \text{if } N = 2, \\ \text{and so on,} & \text{if } N > 2, \end{cases} \]

with \([U(\mathbf{x})\text{ is unitary}]\)

\[ M =: U^\dagger \text{diag}(\lambda_1, \ldots, \lambda_N) U, \]

\[ \text{sgn}[\mathcal{M}(\mathbf{x})] := U^\dagger \text{diag}\left(\frac{\lambda_1}{|\lambda_1|}, \ldots, \frac{\lambda_N}{|\lambda_N|}\right) U. \]
The parameters for the phase diagram: \( \pi_0(V) = \mathbb{Z}_2 \)

For any of the two AZ symmetry classes with \( V = R_1, R_2 \), we can always choose the parameter space \( (m, g) \in \mathbb{R} \times [0, \infty[ \) by selecting

\[
\alpha = \alpha_{\text{min}} \otimes 1_N, \quad \mathcal{M}(x) = \rho_{\text{min}} \otimes M(x),
\]

with \( \rho_{\text{min}} \otimes \sigma_2 \) delivering a representation of the Clifford algebra of rank \( r_{\text{min}} \),

\[
M(x) = M^\dagger(x), \quad M(x) = -M^T(x), \quad m := \text{Pf}[iM(x)],
\]

and

\[
\overline{\mathcal{M}(x)} = 0, \quad \frac{1}{r} \text{tr} \left[ \mathcal{M}(x) \mathcal{M}(y) \right] =: g^2 e^{-|x-y|/\xi_{\text{dis}}}.
\]
Moreover, the condition

$$\det[M(x)] = 0$$

defines the boundaries of the domains for the quantum network model, each of which can be assigned the index

$$(-1)^\nu := \frac{\text{Pf}[iM(x)]}{\sqrt{\det[iM(x)]}}.$$
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<table>
<thead>
<tr>
<th>AZ symmetry class</th>
<th>$r_{\text{min}}$</th>
<th>$V_{d=1,r}$</th>
<th>$\pi_0(V_{d=1,r})$</th>
<th>Phase diagram</th>
<th>Cut at $m = 0$</th>
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<tbody>
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</table>

(a) A, Al, AII, C, CI
(b) AlIII, BDI, CII
(c) D, DIII

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The breakdown of the topological classification
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<th>Cut at ( m = 0 )</th>
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(a) A, C (N=1)  
(b) A, C (N=2)  
(c) D (N=1)  
(d) D (N=2)  
(g) Al, CI

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(a) A, C (N=1)

(b) A, C (N=2)

(g) All

(c) D (N=1)

(d) D (N=2)

(h) AII, CI

(e) DIII (N=1)

(f) DIII (N=2)

(i) All, BDI, CII

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## Application to 3D space

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<tr>
<td>DIII</td>
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<tr>
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<td>4</td>
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<td>$\mathbb{Z}_2$</td>
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<tr>
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<td>Nonsing</td>
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(a) A, Al, BDI, D, C  
(b) AIII, DIII, CI (N=1)  
(c) AIII, DIII, CI (N=2)  
(d) All, CII
Main results
Introduction
Strategy
Examples
Application to the surfaces of SnTe
Summary
Appendices

C. Mudry (PSI)  The breakdown of the topological classification
Summary

- We have considered (noninteracting) random Dirac Hamiltonians for which we have explored the physics of Anderson localization. Dimensionality of space and the AZ symmetry classes (tenfold way) play an essential role.

- Localization requires the existence of random Dirac masses. There are four possibilities in a given AZ symmetry class:
  1. No random Dirac masses are permitted for sufficiently small rank, these are the delocalized boundary states of a topological insulator.
  2. At least two anticommuting Dirac mass matrices are permitted for 5 symmetry classes when the rank is $r_{\text{min}}$. The insulating phase is unique.
  3. A unique (up to a sign) Dirac mass matrix is permitted for 3+2 symmetry classes when the rank is $r_{\text{min}}$.
    - For 3 of the symmetry classes, a unique (up to a sign) Dirac mass matrix is permitted when the rank is $r_{\text{min}} N$ for any $N = 1, 2, 3, \ldots$.
    - For 2 of the symmetry classes, a unique (up to a sign) Dirac mass matrix is permitted iff the rank is $r_{\text{min}} N$ for any $N = 1, 3, 5, \ldots$. 

C. Mudry (PSI)
In any dimension, the density of states can be singular at the band center in 5 out of the 10 AZ symmetry classes:

- In any dimension, the three chiral classes always support point defects in their Dirac masses labeled by a topological index in $\mathbb{Z}$.
- In any dimension, 2 of the remaining 7 symmetry classes support point defects in their Dirac masses labeled by a topological index in $\mathbb{Z}_2$.

Case $d = 1$

Five AZ symmetry classes

(A)

Three AZ symmetry classes

(B) $N=1$

Two AZ symmetry classes

(C) $N=2$ and so on

\begin{align*}
\nu = 0 & & \nu = 0 & & \nu = +1 \\
\nu = -1/2 & & \nu = +1/2 & & \nu = -1/2
\end{align*}
Case $d = 3$

Five AZ symmetry classes

Three AZ symmetry classes

Two AZ symmetry classes

(A) $N=1$

(B) $N=2$ and so on

(C)

C. Mudry (PSI)