

*Quantum Phase Transitions between Bosonic
Symmetry Protected Topological States without
sign problem*

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Quantum Phase Transitions between bosonic Symmetry Protected Topological States without sign problem

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Collaborators from China:

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Dan Mao;

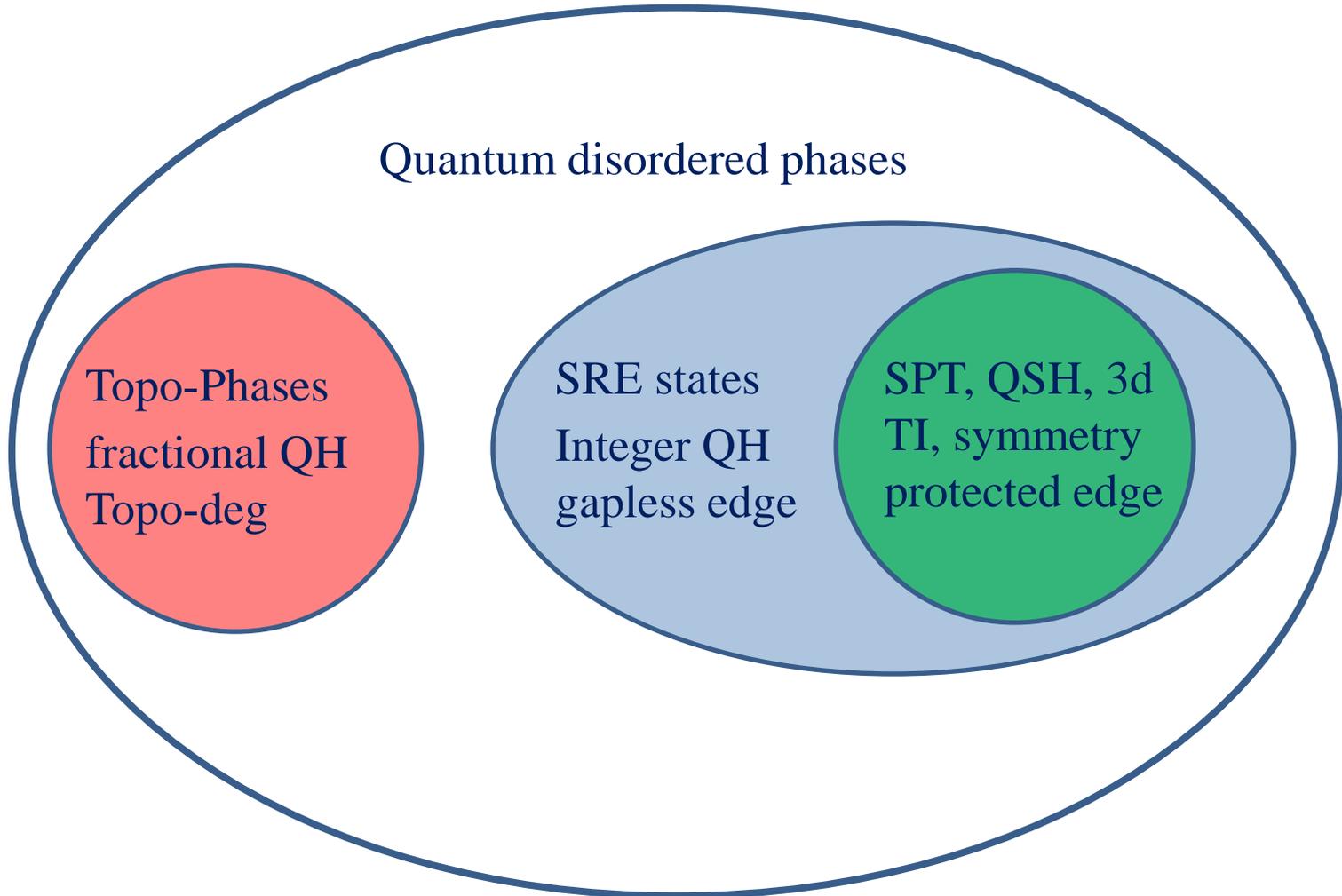
Main references:

Slagle, You, Xu, arXiv:1409.7401,

He, Wu, You, Xu, Meng, Lu, arXiv:1508.06389

You, Bi, Mao, Xu, arXiv:1510.04278

“All” the quantum disordered phases:



Oversimplified Introduction to SPT states:

Symmetry Protected Topological States (oversimplified introduction):

Free fermion version:

d-dimensional bulk: massive Dirac/Majorana fermion;

(d-1)-dimensional boundary: gapless Dirac/Weyl/Majorana fermions, gapless spectrum protected by symmetry, i.e. Symmetry forbids fermion mass term.

Full classification: S. Ryu, et.al. 2009, A. Kitaev 2009;

Phase transition between free fermion SPT states:

Gapless Dirac/Majorana fermion in the d-dimensional bulk. Simplest example is the IQH plateau transition, described by a single 2+1d Dirac fermion, with mass m tuned to zero:

$$L = \bar{\psi}(i\gamma_{\mu}\partial_{\mu} + m)\psi$$

Oversimplified Introduction to SPT states:

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Full classification: S. Ryu, et.al. 2009, A. Kitaev 2009;

Interacting fermion SPTs:

Interaction can reduce the classification of fermion SPTs,

1d: \mathbb{Z} reduce to \mathbb{Z}_8 , Fidkowski, Kitaev, 2009

2d: \mathbb{Z} reduce to \mathbb{Z}_8 , Qi, 2012, Ryu, 2012.....

3d: \mathbb{Z} reduce to \mathbb{Z}_{16} , He3B, Xie Chen et.al., Chong Wang, et.al. and...

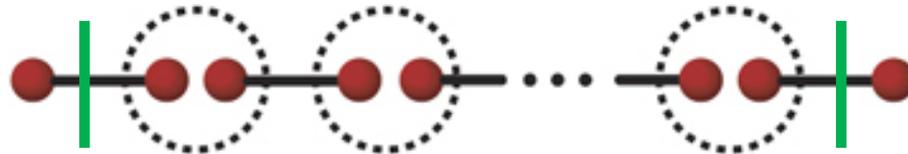
4d: \mathbb{Z} reduce to \mathbb{Z}_{16} , connection to the Standard Model...

Oversimplified Introduction to SPT states:

Symmetry Protected Topological States (oversimplified introduction):

Bosonic SPT states:

There is no free boson version; always strongly interacting;
simplest example; 1d Haldane phase of spin-1 chain:



Field theory description: $O(3)$ NLSM + Θ -term, for $\pi_2[S^2] = \mathbb{Z}$.
Haldane 1988, Ng 1994, Coleman 1976.

$$\mathcal{S} = \int dx d\tau \frac{1}{g} (\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi} \epsilon_{abc} \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c \quad \Theta = 2\pi$$

Field theory built with AF order parameter.

Oversimplified Introduction to SPT states:

Symmetry Protected Topological States (oversimplified introduction):

Bosonic SPT states:

Higher dimensional bosonic SPT states, much more complicated, can be classified (slightly incomplete), **Chen, Gu, Liu, Wen 2011**

Questions:

- (1) Can there be a (quasi-)realistic, and sign problem-free Hamiltonian for higher dimensional bosonic SPT states?
- (2) Can interaction fundamentally change the nature of quantum phase transition between fermionic SPT and trivial states?

Goal of this talk:

Kill the two questions above with one (series of) model.

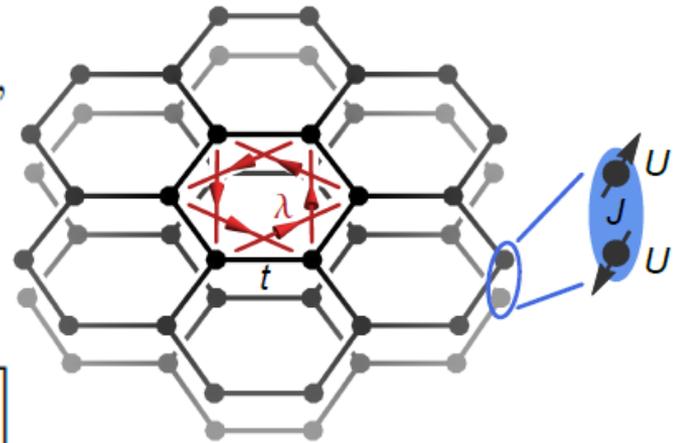
Lattice model: version-1

Version-1 of our model:

$$H = H_{\text{band}} + H_{\text{int}},$$

$$H_{\text{band}} = -t \sum_{\langle ij \rangle, \ell} c_{i\ell}^\dagger c_{j\ell} + \sum_{\langle\langle ij \rangle\rangle, \ell} i\lambda_{ij} c_{i\ell}^\dagger \sigma^z c_{j\ell} + H.c.,$$

$$H_{\text{int}} = \frac{U}{2} \sum_{i, \ell} (n_{i\ell} - 1)^2 + J \sum_i \left[\mathbf{S}_{i1} \cdot \mathbf{S}_{i2} + \frac{1}{4} (n_{i1} - 1)(n_{i2} - 1) - \frac{1}{4} \right]$$



Simple limits of this model:

- (1) Noninteracting: bilayer quantum spin Hall, boundary c=2 CFT;
- (2) Strong J -interacting limit: trivial Mott insulator:

$$|\Psi\rangle = \prod_i (c_{i1\uparrow}^\dagger c_{i2\downarrow}^\dagger - c_{i1\downarrow}^\dagger c_{i2\uparrow}^\dagger) |0\rangle$$

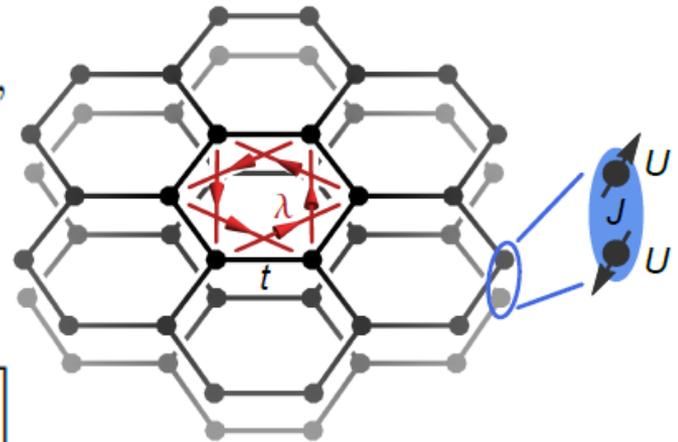
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When $U=0$, this model has an emergent $SO(4)$ symmetry, the $SO(4)$ factorize into $SU(2)_L$ and $SU(2)_R$ at the boundary, namely spin-up and spin-down each has its own $SU(2)$ symmetry

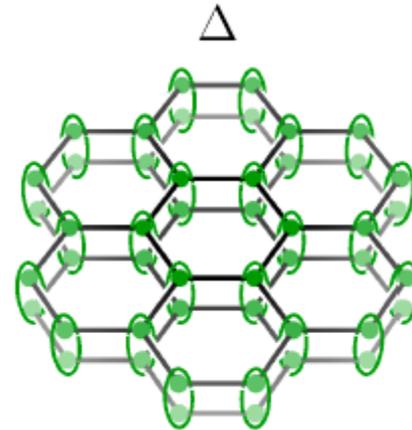
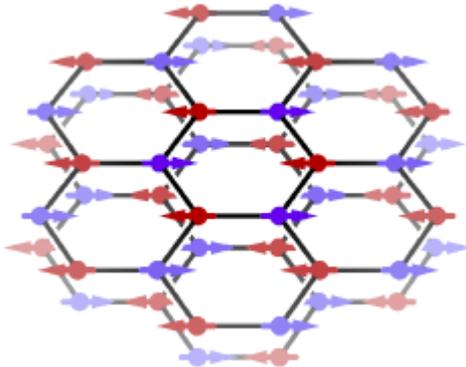
Lattice model: version-1

Version-1 of our model:

When $U=0$, this model has an emergent SO(4) symmetry: SO(4) vector:

$$\begin{aligned} \mathbf{n}_i &= (N_i^x, \text{Im } \Delta_i, \text{Re } \Delta_i, N_i^y). \\ &= f_{i\downarrow}^\dagger (\tau^0, i\tau^1, i\tau^2, i\tau^3) f_{i\uparrow} + h.c., \\ &\quad (N^x, N^y) \end{aligned}$$

$$f_{i\uparrow} = \begin{pmatrix} c_{1i\uparrow} \\ (-1)^i c_{2i\uparrow}^\dagger \end{pmatrix}, \quad f_{i\downarrow} = \begin{pmatrix} (-1)^i c_{1i\downarrow} \\ c_{2i\downarrow}^\dagger \end{pmatrix}$$



$$H = \sum_{i,j,\sigma} (-)^\sigma f_{i\sigma}^\dagger (-t_{ij} + i\lambda_{ij}) f_{j\sigma} + h.c. - \frac{J}{16} \sum_i (D_i D_i^\dagger + D_i^\dagger D_i) \quad D_i = \sum_\sigma f_{i\sigma} i\tau^2 f_{i\sigma}$$

Lattice model: version-1

Boundary analysis:

(1) Can be understood by direct bosonization: **Interaction can gap out all the fermion modes**, and drive the boundary into a $c=1$ CFT, which corresponds to $SU(2)_1$ CFT, with gapless $SO(4)$ vector:

$$\mathbf{n}_i = (N_i^x, \text{Im } \Delta_i, \text{Re } \Delta_i, N_i^y).$$

(2) Integrating out the boundary fermions, it generates the following $SO(4)$ NLSM with WZW term at level-1, which flows to $c=1$ $SU(2)_1$ CFT in the infrared limit:

$$S = \int d\tau dx du \frac{1}{2g} (\partial_\mu \mathbf{n})^2 + \frac{ik}{2\pi} \epsilon_{abcd} n^a \partial_\tau n^b \partial_x n^c \partial_u n^d$$

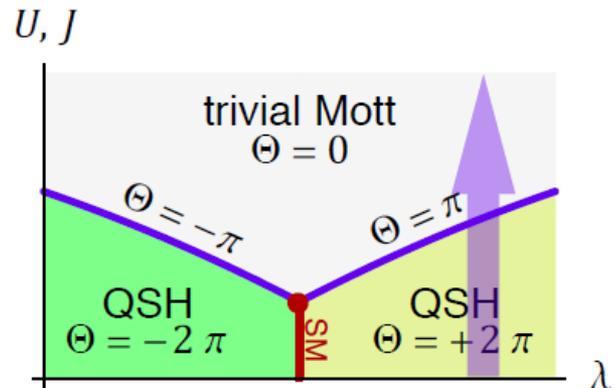
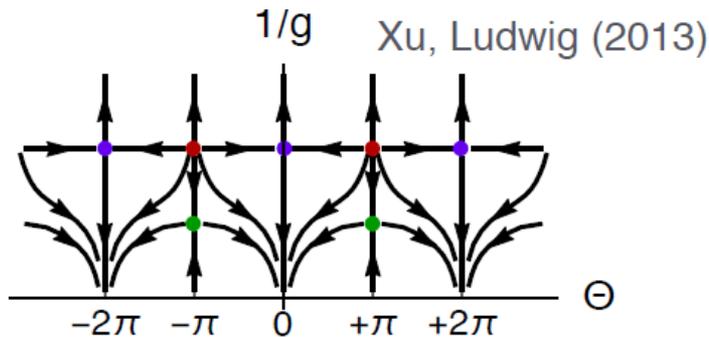
Lattice model: version-1

Bulk analysis:

Harder, but knowing the boundary theory is the purely bosonic SO(4) NLSM with WZW term at level-1, the bulk theory is:

$$S = \int d^2x d\tau \frac{1}{g} (\partial_\mu n)^2 + \frac{i\Theta}{\Omega_3} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d$$

With stable fixed point $\Theta = 2\pi$ describes the bilayer QSH with interaction, which now is equivalent to a bosonic SPT phase, and stable fixed point $\Theta = 0$ describes the trivial Mott insulator. The phase transition corresponds to $\Theta = \pi$.



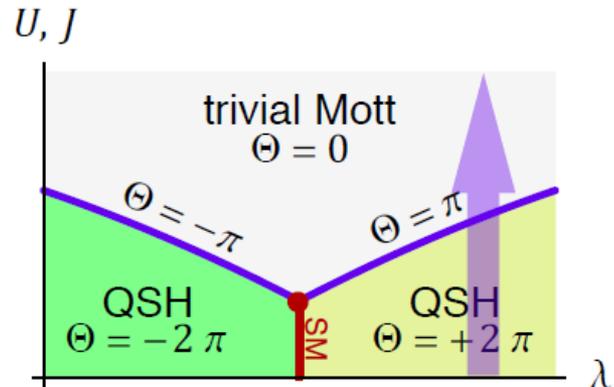
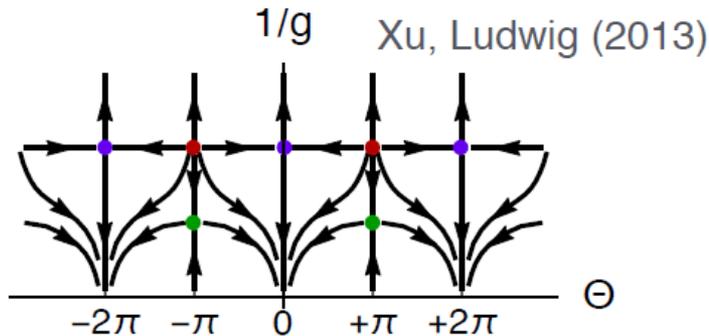
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Our lattice model gives us a way to simulate this field theory with topological term without sign problem. Because boundary has no gapless fermion mode, we expect bulk quantum phase transition also only has bosonic gapless modes.

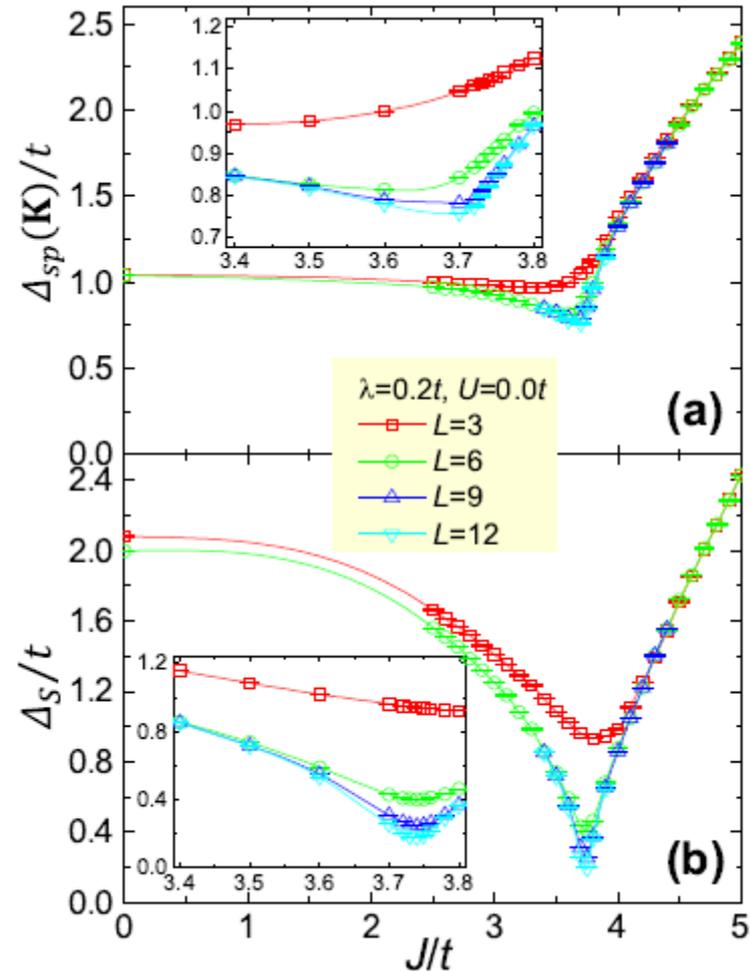


Lattice model: version-1

Bulk analysis:

Indeed, our determinant QMC data ([arXiv:1508.06389](https://arxiv.org/abs/1508.06389)) shows that the single fermion particle gap is finite, but bosonic modes, the $SO(4)$ vector \mathbf{n} , becomes gapless at the SPT-trivial Quantum critical point. This is very different from free fermion phase transition.

Because the fermion degree of freedom never shows up at either the boundary or the bulk transition, this is also a bosonic SPT transition.



Lattice model: version-1

Remarks

(1) The O(4) NLSM with $\Theta = 2\pi$ can describe all the (2+1)d bosonic SPTs with physically relevant symmetries (any combination of U(1), Z₂, time-reversal, etc.), this field theory gives the correct boundary states, and bulk wave function:

Xu, Senthil, 2013, Bi, Rasmussen, Xu, 2013

$$S = \int d^2x d\tau \frac{1}{g} (\partial_\mu n)^2 + \frac{i\Theta}{\Omega_3} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d$$

(2) The same model O(4) NLSM with $\Theta = \pi$ can describe all the (2+1)d boundary of many (3+1)d bosonic TI (Vishwanath, Senthil, 2012). Here $\Theta = \pi$ is guaranteed by symmetry, cannot be freely tuned.

(3) The 3d BTI boundary can also be QED₃ with $N_f=2$, and this theory is **self-dual**. (Xu, You, arXiv:1510.xxxxx)

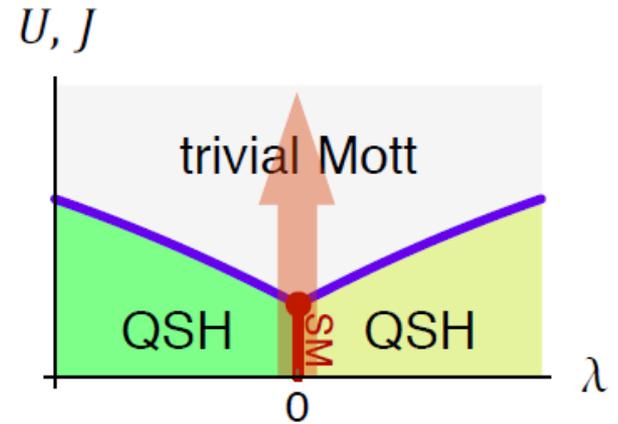
Lattice model: version-1

Other features of this model:

$$H = H_{\text{band}} + H_{\text{int}},$$

$$H_{\text{band}} = -t \sum_{\langle ij \rangle, \ell} c_{i\ell}^\dagger c_{j\ell} + \sum_{\langle\langle ij \rangle\rangle, \ell} i\lambda_{ij} c_{i\ell}^\dagger \sigma^z c_{j\ell} + H.c.,$$

$$H_{\text{int}} = \frac{U}{2} \sum_{i, \ell} (n_{i\ell} - 1)^2 + J \sum_i \left[\mathbf{S}_{i1} \cdot \mathbf{S}_{i2} + \frac{1}{4} (n_{i1} - 1)(n_{i2} - 1) - \frac{1}{4} \right]$$

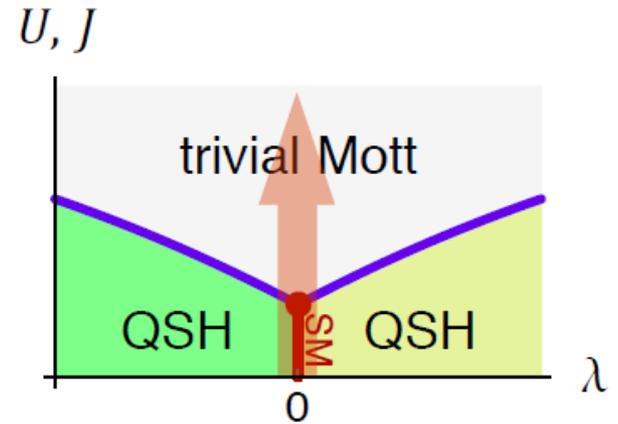
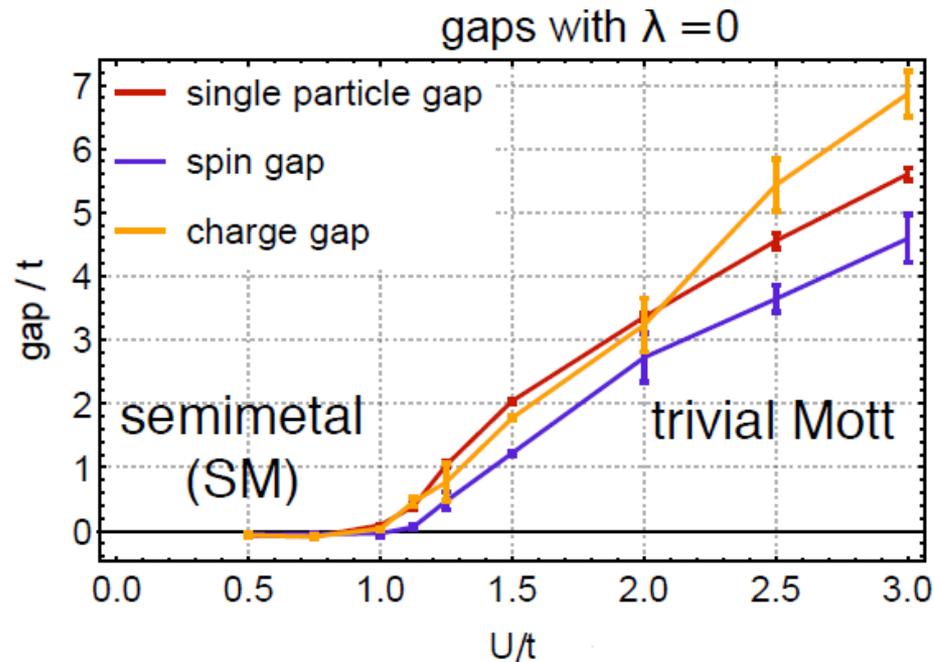


Slagle, You, Xu
arXiv:1409.7401

With $\lambda = 0$, $J=2U$, this model has a $\text{SO}(5)$ symmetry, we are interested in how the semi-metal to trivial Mott insulator phase transition happen.

Lattice model: version-1

Other features of this model:



Slagle, You, Xu
arXiv:1409.7401

In our finite size system, the single particle gap, boson gap, seem to open up at the same critical point, without generating any fermion bilinear mass operator. This is very different from the standard interaction driven mass generation for Dirac fermion.

Lattice model: version-1

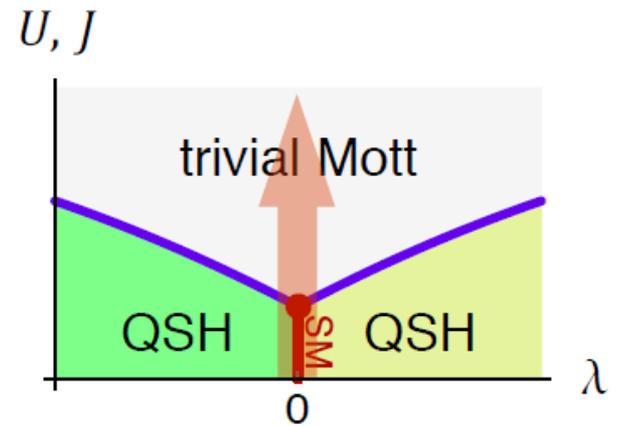
Other features of this model:

Total number of Dirac cones: $8 = 2 \times 2 \times 2$,
Total number of Majorana fermion cones:
 $16 = 2 \times 2 \times 2 \times 2$;

Interaction can reduce the classification of He3B from Z to Z_{16} , namely interaction can gap out 16 copies of Majorana fermions without generating any fermion bilinear mass term (forbidden by symmetry). Our result suggests that this “anomalous mass generation” transition can be continuous.

Same results (with the same counting of Dirac/Majorana fermions) were obtained in lattice QCD community.

Chandrasekharan: arXiv:1412.3532, Catterall, arXiv:1510.04153



Slagle, You, Xu
arXiv:1409.7401

Lattice model: version-N

Generalize our 2d lattice model to larger-N

$$H = H_{\text{band}} + H_{\text{int}}$$

$$H_{\text{band}} = -t \sum_{\langle ij \rangle, \ell} c_{i\ell}^\dagger c_{j\ell} + \sum_{\langle\langle ij \rangle\rangle, \ell} i\lambda_{ij} c_{i\ell}^\dagger \sigma^z c_{j\ell} + H.c.$$

$$H_{\text{int}} = -U \sum_i M_i \cdot M_i,$$

$$M_i^- = 2(-)^i \sum_{\ell \in \text{odd}} c_{i,\ell} c_{i,\ell+1}^\dagger, \quad M_i^3 = \sum_{\ell} (-)^{i+\ell} c_{i\ell}^\dagger \sigma^z c_{i\ell}$$

The entire model has $\text{Sp}(N) \times \text{Sp}(N)$ symmetry, spin up and spin down both have its own $\text{Sp}(N)$ symmetry.

In the noninteracting limit, the system is N -copies of QSH insulator, its 1+1d boundary is non chiral CFT with central charge $c=N$.

Lattice model: version-N

Boundary analysis:

CFT decomposition for 1+1d free fermions:

$$U(2N)_1 \simeq O(4N)_1 \simeq Sp(N)_1 + SU(2)_N.$$

$$c_{Sp(N)_1} = \frac{N(2N+1)}{N+2}, \quad c_{SU(2)_N} = \frac{3N}{N+2},$$

$$:\psi_\sigma^\dagger i\partial_x \psi_\sigma: = \frac{2\pi}{N+2} (J_{Sp(N)_\sigma}^a J_{Sp(N)_\sigma}^a + J_{SU(2)_\sigma}^a J_{SU(2)_\sigma}^a)$$

The interaction we designed, will generate the following term for the boundary theory:

$$H_{\text{int}} = -16\lambda_M J_{SU(2)_R}^a J_{SU(2)_L}^a$$

This interaction is marginally relevant, it will gap out the $SU(2)_N$ part of the boundary, leaves only the $Sp(N)_1$ CFT.

Lattice model: version-N

Bulk analysis:

The interaction we designed, will generate the following term for the boundary theory:

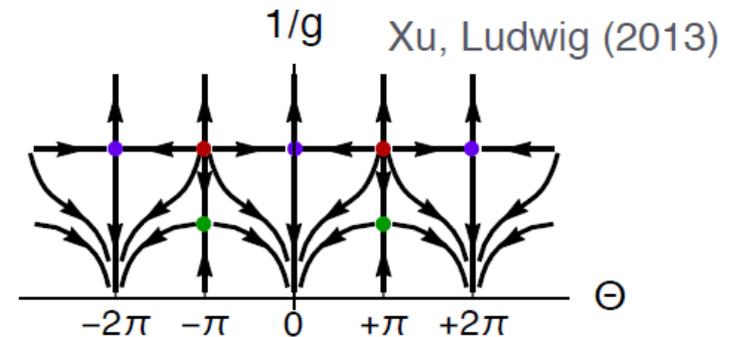
$$H_{\text{int}} = -16\lambda_M J_{\text{SU}(2)_R}^a J_{\text{SU}(2)_L}^a$$

This interaction is marginally relevant, it gaps out the $\text{SU}(2)_N$ part of the boundary (also all fermion modes), leaves only the $\text{Sp}(N)_1$ CFT.

This boundary theory implies that in the weak interacting regime, the bulk is in a bosonic SPT phase, and it is described by the following $\text{Sp}(N)$ principal chiral model with fixed point $\Theta = 2\pi$:

$$S = \int d\tau d^2x \frac{1}{g} \text{tr}[\partial_\mu U^\dagger \partial_\mu U]$$

$$+ \frac{i\Theta}{24\pi^2} \epsilon_{\mu\nu\rho} \text{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)]$$

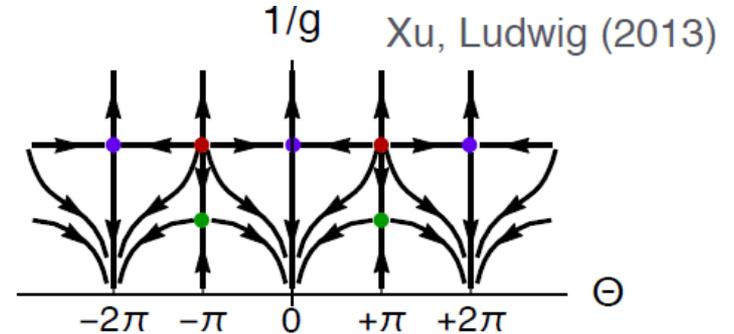


Lattice model: version-N

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$$+ \frac{i\Theta}{24\pi^2} \epsilon_{\mu\nu\rho} \text{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)]$$



In the lattice model, one can prove that by increasing the interaction U , there will be a transition between SPT and trivial Mott insulator.

$$H_{\text{band}} = -t \sum_{\langle ij \rangle, \ell} c_{i\ell}^\dagger c_{j\ell} + \sum_{\langle\langle ij \rangle\rangle, \ell} i\lambda_{ij} c_{i\ell}^\dagger \sigma^z c_{j\ell} + H.c.$$

$$H_{\text{int}} = -U \sum_i M_i \cdot M_i,$$

This transition happens at $\Theta = \pi$ in the field theory. Again this model can be simulated without sign problem.

Lattice model: version-N

Summary and Generalizations:

We designed a series of lattice fermion models with short range interactions without sign problem, that has a novel phase transition between a bosonic SPT phase and a trivial Mott insulator phase. The fermions are always gapped at the boundary and also at the bulk phase transition. This is very different from any free fermion TI phase transition.

By generalizing these models, we may find another set of novel QCPs for the transition between $(2+1)d$ CFT and the ordered phase with SSB.

