ANOMALOUS HYDRODYNAMICS OF VORTEX FLOW IN TWO-DIMENSIONAL FLUID

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2D incompressible flows consist of vortices
HYDRODYNAMICS OF VORTEX FLUID

Vortices as constituencies of a secondary fluid - the vortex fluid (or vortex matter)

- Fast motion: fluid precessing around vortices;
- Slow motion of vortices.
- **What is the hydrodynamics of vortex fluid?**
Euler Hydrodynamics $\Rightarrow$ Anomalous hydrodynamics
Outline

- Hydrodynamics: Incompressible flows in two dimensions
- Search for Conformal Invariance
- Kirchhoff equations
- Onsager ensemble and Random Matrix Theory
- Anomalous forces in hydrodynamics
- Hydrodynamics of vortex flow, superfluids and FQHE
Hydrodynamics of the vortex flow is anomalous

Assumption: Circulations of vortices are bounded $> \Gamma$
ANOMALOUS HYDRODYNAMICS

Euler Equation

\[ D_t \mathbf{u} = -\nabla p \]

Anomalous Euler Equation

\[ D_t v_a = -\nabla_a p + \nabla_b \tau_{ab} \]

\[ D_t \equiv \partial_t + \mathbf{u} \cdot \nabla \text{ - Material Derivative.} \]

\( \tau_{ab} \) – anomalous stress - symmetric pseudo-tensor

2D:

\[ \tau_{xy} = \tau_{yx} = -\eta (\nabla_x u_x - \nabla_y u_y), \]

\[ \tau_{xx} = -\tau_{yy} = \eta (\nabla_x u_y + \nabla_y u_x), \]

\[ \tau = \tau_{xx} - \tau_{yy} - 2i\tau_{xy} = -2i\eta \partial u, \quad \tau_{zz} = 0 \]

\[ \eta \] – a universal anomalous kinetic coefficient
Anomalous Hydrodynamics
HYDRODYNAMICS OF INCOMPRESSIBLE FORCES

Euler Equation

\[ D_t u = -\nabla p, \]

Material Derivative

\[ D_t \equiv (\partial_t + u \cdot \nabla) \]

Incompressibility

\[ \nabla \cdot u = 0, \]

Vorticity

\[ \varpi = \nabla \times u \]

Vorticity is transported along the velocity field: the material derivative of the vorticity in that flow vanishes:

\[ \frac{D\varpi}{Dt} \equiv \dot{\varpi} + u \cdot \nabla \varpi = 0. \]
Kirchhoff Equations

\[ \frac{D\varpi}{Dt} \equiv \dot{\varpi} + \mathbf{u} \cdot \nabla \varpi = 0. \]

Helmholtz (and later Kirchhoff)

\[ u(z, t) = u_x - iu_y = i \sum_{i=1}^{N} \frac{\Gamma_i}{z - z_i(t)}. \]

Kirchhoff equations

\[ i \dot{z}_i = \sum_{i \neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)}. \]
Chiral Flow: Clustering, Rotating Fluid

Chiral Kirchhoff equations $\Gamma_i = \Gamma$

$$i\dot{\mathbf{z}}_i = \Omega\mathbf{z}_i - \sum_{i \neq j}^N \frac{\Gamma}{z_i(t) - z_j(t)}$$

Object of interest: Large $N$ limit such that the area of the patch is fixed
Kirchhoff equations are Hamiltonian

- Poisson brackets

\[ \{ z_i, \tilde{z}_j \}_{P.B.} = (i\pi \Gamma)^{-1} \delta_{ij}. \]

- Hamiltonian

\[ \mathcal{H} = \Omega |z_i|^2 - \Gamma^2 \sum_{j \neq i} \log |z_i - z_j|^2 \]
Onsager ensemble: Stochastic Hydrodynamics

Thermodynamics of the vortex gas

\[ \mathcal{P}(z_1, \ldots, z_N) = \prod_{i \neq j} |z_i - z_j|^{2\beta} e^{-\sum_i |z_i|^2/4\ell^2}, \quad 2\beta = \Gamma^2 / T \]
HYDRODYNAMICS OF ONSAGER FLUID OF VORTICES

○ Start from the many body system

\[ \dot{\mathbf{z}}_i = \mathbf{v}_i = \Omega \mathbf{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{\mathbf{z}_i(t) - \mathbf{z}_j(t)} \]

○ Reformulate it through the density

\[ \rho(r) = \sum_{i} \delta(r - r_i) = (2\pi \Gamma)^{-1} \omega(r). \]

and velocity

\[ J = \rho(r)v(r) \equiv \sum_{i} \delta(r - r_i)v_i, \]

○ write evolution equations for density \( \rho \) and velocity \( \mathbf{v} \)

\[ \mathcal{D}_t \rho = \ldots, \quad \mathcal{D}_t \mathbf{v} = \ldots \]

○ Compare with the Euler equations

\[ D_t \rho = 0, \quad D_t \mathbf{u} = -\nabla p \]
CHIRAL RELATION

In the chiral flow position of vortices determines their velocity:

\[ i \dot{z}_i = v_i = \Omega z_i - \sum_{i \neq j}^N \frac{\Gamma}{z_i(t) - z_j(t)} = \frac{1}{i\pi \Gamma} \partial_{z_i} \mathcal{H} \]

\[ \rho \leftrightarrow v \]

\[ v(z) = \frac{1}{i\pi \Gamma} \frac{\partial \mathcal{H}}{\partial \rho(z)} \]
OBJECTS IN HYDRODYNAMICS

- Flux \[ J = \rho \, v = (\pi \Gamma) \rho \, v = i \bar{\partial} \mathcal{T} \]

- Stress tensor: a response of the energy to a general transformation of coordinates and dilatations
  \[ z \to z + \epsilon(z, \bar{z}), \quad \text{and dilatations} \quad \rho \to \rho + \lambda(z, \bar{z}) \rho \]

\[ \mathcal{T}_{z\bar{z}}(z) = -\rho(z) \frac{\delta \mathcal{H}}{\delta \rho(z)}, \]

\[ \mathcal{T}(z) = \frac{1}{\pi} \sum_i \frac{1}{z - z_i} \frac{\partial \mathcal{H}}{\partial z_i} = i\Gamma \sum_i \frac{v_i}{z - z_i}. \]

\[ \bar{\partial} \mathcal{T} + \rho \partial (\rho^{-1} \mathcal{T}_{z\bar{z}}) = 0 \]
**Stress tensor in Anomalous Hydrodynamics**

Stress tensor in Euler hydrodynamics (the holomorphic component only):

\[
\mathcal{T} = \frac{1}{2} u^2 = \frac{1}{2} (\partial \psi)^2
\]

Stress tensor in vortex (anomalous) hydrodynamics:

\[
\mathcal{T} = \frac{1}{2} u^2 - i \frac{\Gamma}{2} \partial u = \frac{1}{2} (\partial \psi)^2 - i \frac{\Gamma}{2} \partial^2 \psi
\]

(anomalous term)

\[\psi - \text{stream function: } u_x = -\nabla_y \psi, \quad u_y = \nabla_x \psi\]
**Calculations**

We want to express

\[ \mathcal{T}(z) = i \Gamma \sum_i \frac{v_i}{z - z_i}, \quad v_i = \sum_{i \neq j} \frac{\Gamma}{z_i - z_j} \]

Through velocity

\[ u = \sum_j \frac{\Gamma}{z - z_j} \]

Use the identity

\[ 2 \sum_{i \neq j} \frac{1}{z - z_i} \frac{1}{z_i - z_j} = \left( \sum \frac{1}{z - z_i} \right)^2 + \partial \left( \sum \frac{1}{z - z_i} \right) \]

To obtain

\[ \mathcal{T} = \frac{1}{2} u^2 - i \frac{\Gamma}{2} \partial u \]
DEFLECTION OF THE VELOCITY AND STREAM LINES

Anomalous term in the velocity

\[ \rho(r)v(r) = \sum_i \delta(r - r_i)v_i, \quad v_i = \sum_{i \neq j}^N \frac{\Gamma}{z_i - z_j} \]

\[ v = u + \frac{\Gamma}{2} i \partial \log |\omega| \]

\[ \omega = \nabla \times u \]

and in terms of the stream lines

\[ \Psi = \psi + \frac{\Gamma}{4} \log \Delta \psi \]

\[ v = -\nabla \times \Psi, \quad u = -\nabla \times \psi \]
HAMILTONIAN AND POISSON ALGEBRA OF THE VORTEX FLOW

\[ \mathcal{H} = \frac{1}{2} \int \left[ v^2 - \left( \frac{\Gamma}{4} \nabla \log \rho \right)^2 \right] d^2 r, \]

The chiral constraint:

\[ (2\pi \Gamma) \cdot (\nabla \times v) = \rho + \frac{\Gamma}{4} \Delta \log \rho. \]

Poisson algebra \( J = \rho v \)

\[ \{ \rho(r), \rho(r') \} = -\pi \Gamma (\nabla_r \times \nabla_{r'}) [((\rho(r) + \rho(r')) \delta(r-r')], \]

\[ \{ \tilde{J}(r), J(r') \} = \left( -\frac{1}{2} (J \times \nabla) + \pi \Gamma \left( \rho^2 + \frac{1}{4} \nabla \rho \cdot \nabla \right) \right) \rho \delta(r-r') \]
HYDRODYNAMICS IN A CURVED SPACE: TRACE ANOMALY

Riemann manifold with a metric $g_{ab}$: $\rho \rightarrow \rho \sqrt{g}$ The energy

$$H \rightarrow H + \frac{\Gamma^2}{32} \int R \Delta_g^{-1} R dV$$

Liouville action

The stress tensor

$$\mathcal{T}_{\bar{z}z} \rightarrow \mathcal{T}_{\bar{z}z} - \frac{\Gamma^2}{4} R$$

Density in a stationary flow

$$\delta \rho = \frac{1}{8\pi} R$$

A cone with a deficit angle $\theta$ accumulates $\theta / 4\pi$ vortices.
Kirchhoff Equations are Hamiltonian and finite dimensional - readily for quantization;

\[ i\dot{z}_i = \sum_{i \neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)} \]

\( \{z_i, \bar{z}_j\}_{P.B.} \rightarrow [z_i, \bar{z}_j] = \beta^{-1} \delta_{ij}, \quad \Gamma = \beta \hbar. \)

\( \bar{z}_i \rightarrow \beta^{-1} \partial_{z_i} \)

Attempts to quantize incompressible Euler’s equation failed.
The ground state of the vortex flow is Feynman wave function.

In the chiral case (all vortices are like-sign) is Laughlin’s wave function

\[ v_i |0 > = 0 \]

\[ \left( \partial_{z_i} + \Omega \bar{z}_i - \sum_{i\neq j}^{N} \frac{\beta}{z_i - z_j} \right) \Psi(z_1, \ldots, z_N) = 0 \]
Omitted subjects

- Relation to CFT
- Application to turbulence (project with Eldad)